DOWNWARD TRANSFERENCE OF MICE AND UNIVERSALITY OF LOCAL CORE MODELS

ANDRÉS EDUARDO CAICEDO AND MARTIN ZEMAN

ABSTRACT. If $\mathbf{M}$ is an inner model and $\omega_2^{\mathbf{M}} = \omega_2$, then every sound mouse projecting to ω and not past $0^\P$ belongs to $\mathbf{M}$. In fact, under the assumption that $0^\P$ does not belong to $\mathbf{M}$, $\mathbf{K}^{\mathbf{M}}\|\omega_2$ is universal for all countable mice in $\mathbf{V}$.

Similarly, if $\delta > \omega_1$ is regular, $(\delta^+)^{\mathbf{M}} = \delta^+$, and in $\mathbf{V}$ there is no proper class inner model with a Woodin cardinal, then $\mathbf{K}^{\mathbf{M}}\|\delta$ is universal for all mice in $\mathbf{V}$ of cardinality less than δ.

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1. Introduction

1.1. **Motivation.** This paper is the second in a series started with Caicedo [Cai10]. The goal is to study the structure of (not necessarily fine structural) inner models of the set theoretic universe $\mathbf{V}$, perhaps in the presence of additional axioms, assuming that there is agreement between (some of) the cardinals of $\mathbf{V}$ and of the inner model.

Although the results in this paper do not concern forcing axioms, it was a recently solved problem in the theory of the proper forcing axiom, PFA, that motivated our results.

It was shown in Veličković [Vel92] that if Martin's maximum, MM, holds, and $\mathbf{M}$ is an inner model that computes ω_2 correctly, then $\mathbb{R} \subset \mathbf{M}$. The argument requires the weak reflection principle to hold, a consequence of both MM and PFA^+. Todorčević [Tod93] showed that the same result holds if, rather than MM, the reflection principle known as Rado's conjecture is assumed.

It is natural to wonder whether PFA suffices as well. A partial result in this direction was shown in Caicedo-Veličković [CV06], namely that if $\mathbf{M}$ is an inner model that computes ω_2 correctly, and *both* $\mathbf{V}$ and $\mathbf{M}$ satisfy the bounded proper forcing axiom, BPFA, then again $\mathbb{R} \subset \mathbf{M}$.

However, it was recently shown by Friedman [Fri11] (see also Neeman [Neea]) that the answer to the PFA question is negative: Using a variant of the method of models as side conditions, Friedman defines a forcing notion that, starting with a supercompact κ, produces an extension where PFA holds and κ becomes ω_2. The generic for the partial order where only the side conditions are considered also collapses κ to size ω_2, but the resulting extension does not contain all the reals.

Assume that PFA holds in $\mathbf{V}$, and that $\mathbf{M}$ is an inner model with the same ω_2. It seems not completely unreasonable to expect that, even though not all reals can be guaranteed to be in $\mathbf{M}$, the reals of $\mathbf{V} \setminus \mathbf{M}$ should be *generic* in some sense. A possible way of formalizing this intuition consists in trying to show that the large cardinal strength coded by reals in $\mathbf{V}$ is also present in $\mathbf{M}$. When approaching this problem, we realized that there may be some ZFC results that the presence of forcing axioms could be hiding.

To illustrate this, we mention a couple of observations:

(1) Assume first that $\mathbf{M}$ is an inner model such that $\mathsf{CAR}^{\mathbf{M}} = \mathsf{CAR}$, where CAR denotes the class of cardinals, and that PFA holds. Then, for example, $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ holds in $\mathbf{M}$ and in all its set generic extensions, by Steel [Ste05]. The point here is that Steel [Ste05] shows that $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ holds in any generic extension of the universe by a forcing of size at most κ, whenever $\square_\kappa$ fails for κ a strong limit singular cardinal. If PFA holds, the claim follows by recalling that PFA contradicts $\square_\kappa$ for all uncountable κ, and that $\square_\kappa$ relativizes up to any outer model where κ and κ^+ are still cardinals.

(2) In fact, just the agreement of cardinals guarantees that a significant large cardinal strength is "transfered down" from $\mathbf{V}$ to $\mathbf{M}$. For example, suppose now that $0^{\P}$ exists (see Schindler [Sch02]), and that $\mathbf{M}$ is an inner model with $\omega_3^{\mathbf{M}} = \omega_3$, but no forcing axiom is assumed in $\mathbf{V}$. Then $0^{\P} \in \mathbf{M}$. This is because, otherwise, weak covering holds in $\mathbf{M}$ with respect to $\mathbf{K}^{\mathbf{M}}$, as shown in Schindler [Sch02], and therefore

$$\mathsf{cf}^{\mathbf{M}}(\omega_2^{+\mathbf{K}^{\mathbf{M}}}) \geq \omega_2.$$

However, $\mathsf{cf}(\omega_2^{+\mathbf{K}^{\mathbf{M}}}) = \omega$, since $0^{\P}$ exists. Since $\mathsf{cf}^{\mathbf{M}}(\omega_2^{+\mathbf{K}^{\mathbf{M}}}) \leq \omega_3^{\mathbf{M}}$, this is a contradiction.

(3) If PFA holds and $\mathbf{M}$ is an inner model that satisfies BPFA and computes ω_2 correctly, then $H_{\omega_2} \subset \mathbf{M}$ by Caicedo-Veličković [CV06], so the setting of Claverie-Schindler [CSa] applies. Therefore, using notation as in Claverie-Schindler [CSa], $\mathbf{M}$ is closed under any mouse operator $\mathcal{M}$ that does not go beyond $M_1^\sharp$.

The moral here is that it seems that we only need to assume a local version of agreement of cardinals rather than a global one to conclude that an inner model $\mathbf{M}$ must absorb any significant large cardinal strength coded by reals in $\mathbf{V}$.

1.2. **Main result.** However, for anything like the argument of the second example to hold, we need to be able to invoke weak covering, which seems to require at least agreement of cardinals up to ω_3. (A different, more serious issue, is that arguing via weak covering will not allow us to reach past Woodin cardinals.)

It is then interesting to note that if $\mathbf{M}$ computes ω_2 correctly, and $0^\sharp$ exists, then $0^\sharp \in \mathbf{M}$, see Friedman [Fri02]. We have started a systematic approach to the question of how far this result can be generalized. Using different techniques from those in Friedman [Fri02], we reprove this result, and our method allows us to show that the same conclusion can be obtained with $0^\sharp$ replaced by stronger mice, see for example Lemma 3 and Corollary 10.

Conjecture 1. *Let r be a 1-small sound (iterable) mouse that projects to ω. Assume that $\mathbf{M}$ is an inner model and that $\omega_2^{\mathbf{M}} = \omega_2$. Then $r \in \mathbf{M}$.*

The expected argument to settle Conjecture 1 is an induction on the mice hierarchy. The mice as in the conjecture are lined up so that, if it fails, then there is a least counterexample r. Comparing r with $\mathbf{K}^{\mathbf{M}}$, one expects to reach the contradiction that r is in fact an initial segment of $\mathbf{K}^{\mathbf{M}}$. Our result is that this is indeed the case under appropriate anti-large cardinal assumptions.

We will consider a proper class inner model $\mathbf{M}$ and the true core model of the universe $\mathbf{M}$; we denote this core model by $\mathbf{K}^{\mathbf{M}}$. Of course, we will only consider situations where $\mathbf{K}^{\mathbf{M}}$ exists, so we need to assume that in $\mathbf{M}$

there is no inner model with a Woodin cardinal. We will actually make the assumption that there is no proper class inner model with a Woodin cardinal in the sense of $\mathbf{V}$. Then proper class extender models of $\mathbf{M}$ are fully iterable in $\mathbf{V}$, which will play an important role in the proof of our main theorem. The proof of this kind of absoluteness of iterability is standard, and we give it in Section 2. We briefly comment in Section 4 on what the situation is under the weaker hypothesis that there is no proper class inner model with a Woodin cardinal in the sense of $\mathbf{M}$.

We remark that, although we restrict ourselves to the situation where there are no Woodin cardinals in inner models, we expect that our universality results can be strengthened to reach beyond this point.

We will be using the core model theory as developed in Steel [Ste96]; this approach uses the core model constructed in a universe below a measurable cardinal, which simplifies the exposition. Thus the universe we will be working with will have the property that its ordinals constitute a measurable cardinal in some longer universe. We believe, but did not check carefully, that our arguments for the core model up to a Woodin cardinal can be carried out – in a more or less straightforward way – in the core model theory developed in Jensen-Steel [JS10] which avoids the use of a measurable cardinal. We have tried to make our exposition as independent of the direct reference to the existence of a measurable cardinal in $\mathbf{V}$ as possible, so that there will be enough heuristic evidence for our belief. Regarding the fine structural aspects, we will use Jensen's Σ^*-fine structure theory as described in Zeman [Zem02] or Welch [Wel10], and use some notation from Zeman [Zem02]; we believe this notation is self-explanatory. Otherwise our notation will be fully compatible with that in Steel [Ste96].

Following Steel [Ste96], when working with the core model up to a Woodin cardinal, Ω will be reserved to denote the measurable/subtle cardinal needed to develop the basic core model theory in the universe we are working in. By $\mathbf{V}$, we will mean the rank-initial segment of the universe up to Ω. The notion "class" will then have the corresponding meaning. When working with the core model up to a strong cardinal we let $\Omega = \mathbf{On}$, that is, the class of all ordinals, let $\mathbf{V}$ be the actual universe, and use the core model theory below $0^\P$, the sharp for a strong cardinal, as described in Zeman [Zem02]. It should be stressed that the proof of our main theorem, Theorem 5, will only require to construct $\mathbf{K}^{\mathbf{M}}$, the core model of $\mathbf{M}$. In this case the universe we will be working in will be $\mathbf{M}$, so we will work below Ω relative to $\mathbf{M}$.

Recall the following notation that will be used throughout the paper. If N is any extender model and $\alpha \leq \mathsf{ht}(N)$ is an ordinal, then $N\,||\,\alpha$ is the extender model $\langle J^E_\alpha, E_{\omega\alpha}\rangle$ where E is the extender sequence of N. If α does not index an extender in N, for instance when α is a cardinal in N, then $E_{\omega\alpha} = \varnothing$ and in this case we identify $N\,||\,\alpha$ with J^E_α. Recall also that if $\mathcal{T}$ is an iteration tree of limit length, then $\delta(\mathcal{T})$ is the supremum of all lengths of the extenders used in $\mathcal{T}$ (see also the introductory paragraphs in Section 2).

If $\mathcal{T}$ is an iteration tree of successor length then the main branch of the tree is the branch connecting the first model of the tree with its last model.

The following lemma is a particular case of a well-known result of Shelah, see for example Abraham [Abr83]. The point of this short detour is to explain why we introduce the set S_δ below and concentrate on it rather than directly assuming agreement of cardinals, as discussed above.

Lemma 2. *Let* $\mathbf{M}$ *be a proper class inner model, and suppose that* $\omega_2^{\mathbf{M}} = \omega_2$. *Then* $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ *is stationary.*

Proof. Let $F : [\omega_2]^{<\omega} \to \omega_2$, and pick $\gamma < \omega_2$ of size ω_1 and closed under F. Since $\mathbf{M}$ computes ω_2 correctly, it sees a bijection between the true ω_1 and γ. Then club many countable subsets of γ are both in $\mathbf{M}$ and closed under F. $\square$

To illustrate how the assumption of stationarity of $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ can be used, consider the following:

Lemma 3. *Let* $\mathbf{M}$ *be a proper class inner model, and suppose that* $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ *is stationary. Suppose that* $0^\sharp$ *exists. Then* $0^\sharp \in \mathbf{M}$.

Proof. Assume P is the (unique, sound) mouse representing $0^\sharp$, so

$$P = (J_\tau, \in, U),$$

where U is an amenable measure on the largest cardinal of J_τ, and P has first projectum $\varrho_P^1 = \omega$. Let θ be regular and large. By assumption, we can find some countable $X \preceq H_\theta$ such that:

(1) $P \in X$,
(2) $X \cap \omega_2 \in \mathbf{M}$, and
(3) $\kappa = X \cap \omega_1 \in \omega_1$.

Let H be the transitive collapse of X, and let

$$\sigma : H \to H_\theta$$

be the inverse of the collapsing map. Then the critical point of σ is κ, $\sigma(\kappa) = \omega_1$, and $P \in H$.

Let P' be the κ-th iterate of P, formed by applying the ultrapower construction using U and its images under the corresponding embeddings. Then $P' \in H$ and also

$$\mathcal{P}(\kappa) \cap \mathbf{L} \in H \cap \mathbf{L}.$$

Note that $P' = (J_{\tau'}, U')$, where τ' is the cardinal successor of $\mathsf{cr}(U')$ in $\mathbf{L}$, and that $\sigma \restriction J_{\tau'} \in \mathbf{M}$.

By the argument from the proof of the comparison lemma,

$$\sigma \restriction P' = \pi_{\kappa,\omega_1},$$

where the map on the right side is the iteration map. Hence U' is the $\mathbf{L}$-measure on κ derived from $\sigma \restriction J_{\tau'}$. It follows that $U' \in N$. But then $P \in N$. $\square$

Of course, Lemma 2 generalizes straightforwardly to larger cardinals, which suggests that rather than looking at direct generalizations of Lemma 3 to stronger countable mice, one should instead focus on instances of *local universality.*

Definition 4. *Let δ be a cardinal, or $\delta = \mathbf{On}$.*

An extender model W is **universal** *for extender models of size less than δ if and only if W does not lose the coiteration with any extender model N of size less than δ, that is: Let $(\mathcal{T},\mathcal{U})$ be the pair of iteration trees corresponding to the terminal coiteration of W with N, where $\mathcal{T}$ is on W and $\mathcal{U}$ is on N. Then $\mathcal{T},\mathcal{U}$ are of length less than δ, there is no truncation on the main branch of $\mathcal{U}$, and the last model of $\mathcal{U}$ is an initial segment of the last model of $\mathcal{T}$.*

We also fix the following notation. Given a regular cardinal δ, we let

$$S_\delta = \{x \in \mathcal{P}_\delta(\delta^+) \cap \mathbf{M} \mid \mathsf{cf}^{\mathbf{M}}(x \cap \delta) > \omega\}. \tag{1}$$

Here the cardinal successor δ^+ is computed in $\mathbf{V}$. Note that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ and S_δ are elements of $\mathbf{M}$. Using the terminology from the definition above we can state our main theorem:

Theorem 5. *Assume that $\mathbf{M}$ is a proper class inner model, and that δ is a regular cardinal in the sense of $\mathbf{V}$.*

(a) *Granting that, in $\mathbf{V}$, there is no proper class inner model with a Woodin cardinal, $\delta > \omega_1$, and S_δ is stationary, then the initial segment $\mathbf{K}^{\mathbf{M}} \,||\, \delta$ is universal for all iterable 1-small premice in $\mathbf{V}$ of cardinality less than δ.*

(b) *Granting that, in $\mathbf{M}$, $0^{\P}$ does not exist, and $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary, then: If $\delta > \omega_1$, then the initial segment $\mathbf{K}^{\mathbf{M}} \,||\, \delta$ is universal for all iterable premice in $\mathbf{V}$ of cardinality less than δ. Moreover, if $\delta \geq \omega_1$, then $\mathbf{K}^{\mathbf{M}} \,||\, \omega_2$ is universal for all countable iterable premice in $\mathbf{V}$.*

The proof of Theorem 5 is the heart of this paper and occupies Section 3. We believe that the anti-large cardinal hypothesis in (b) can be weakened to that in (a), that is, "no inner model with a Woodin cardinal" in the respective universe (but note that the anti-large cardinal assumption in (b) only concerns $\mathbf{M}$ rather than $\mathbf{V}$). We also believe that the assumption on S_δ being stationary in (a) can be weakened to the weaker assumption in (b) that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. Unfortunately, as of now we do not see arguments that could be run under these weaker hypotheses. On the other hand, the conclusion in (b) that $\mathbf{K}^{\mathbf{M}} \,||\, \omega_2$ is universal for countable mice in $\mathbf{V}$ cannot be improved by replacing ω_2 by ω_1. In fact, such conclusion is false even if $\mathbf{M} = \mathbf{V}$, by a result of Jensen [Jena]. For example, it is consistent (under mild large cardinal assumptions) that there is a countable mouse m that iterates past $\mathbf{K}_{\omega_1}$. In fact, starting with $\mathbf{K} = \mathbf{L}[U]$ for $U = E_\nu$ a measure, so $\nu = \kappa^{+\mathbf{K}}$ for some κ, Jensen [Jena] describes a forcing extension of $\mathbf{K}$

that turns κ into ω_1, and adds a countable mouse m such that (J^E_ν, E_ν) is an iterate of m.

In Section 4 we indicate what we can prove of Theorem 5.(a) if, instead of the stationarity of S_δ, we only assume that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary, or if instead of the non-existence of inner models with Woodin cardinals, we only require that there are no such inner models from the point of view of $\mathbf{M}$.

1.3. **Applications.**

Corollary 6. *Assume that* $\mathbf{M}$ *is a proper class inner model.*

(a) *Granting that, in* $\mathbf{V}$*, there is no proper class inner model with a Woodin cardinal, and* S_δ *is stationary for proper class many cardinals* δ *that are regular in* $\mathbf{V}$*, then the model* $\mathbf{K}^{\mathbf{M}}$ *is universal for all set sized iterable* 1*-small premice in* $\mathbf{V}$*.*

(b) *Under the assumptions that, in* $\mathbf{V}$*, there is no proper class inner model with a strong cardinal, and* $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ *is stationary for proper class many cardinals* δ *that are regular in* $\mathbf{V}$*, we conclude that* $\mathbf{K}^{\mathbf{M}}$ *is universal with respect to all extender models in* $\mathbf{V}$ *so, in particular, the models* $\mathbf{K}^{\mathbf{M}}$ *and* $\mathbf{K}$ *are Dodd-Jensen equivalent. Here,* $\mathbf{K}$ *is true core model constructed in* $\mathbf{V}$*.*

Proof. The conclusions in (a) are obvious consequences of Theorem 5. The conclusion on the universality of $\mathbf{K}^{\mathbf{M}}$ in (b) follows from the well-known fact that in the absence of proper class inner models with strong cardinals, the universality with respect to set-sized extender models implies the universality with respect to proper class extender models (see, for example, Zeman [Zem02, Lemma 6.6.6]). Since both $\mathbf{K}^{\mathbf{M}}$ and $\mathbf{K}$ are both universal in the sense of $\mathbf{V}$, they are Dodd-Jensen equivalent. □

If there is no proper class inner model with a strong cardinal then the above corollary is a generalization of the well-known fact that $\mathbf{K}$ is generically absolute. In the case of generic absoluteness we let $\mathbf{V}$ play the role of $\mathbf{M}[G]$ where G is a generic filter for some poset in $\mathbf{M}$, and conclude that $\mathbf{K}^{\mathbf{M}} = \mathbf{K}$. If $\mathbf{V}$ is not a generic extension of $\mathbf{M}$ we of course cannot expect that the core model will remain unchanged, but by the above corollary we still can expect that the core models of $\mathbf{M}$ and $\mathbf{V}$ are close.

A well-known example which illustrates the situation where $\mathbf{V}$ is not a generic extension of $\mathbf{M}$ is the following: The model $\mathbf{M}$ is obtained as an ultrapower of $\mathbf{V}$ via some extender in $\mathbf{V}$. Then $\mathbf{K}^{\mathbf{M}}$ is distinct from $\mathbf{K}$, yet $\mathbf{K}^{\mathbf{M}}$ is a normal iterate of $\mathbf{K}$; notice that the hypothesis of the above corollary is satisfied here, as δ can be taken to be the cardinal successor of any singular strong limit cardinal of sufficiently high cofinality. In this case $\delta^{+\mathbf{M}} = \delta^+$ which guarantees that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary in $\mathbf{V}$. Thus, the situation just described is an instance of Corollary 6.

It is well-known that if there is an inner model with a strong cardinal then the conclusion in Corollary 6.(b) is no longer true, that is, it may

happen that $\mathbf{K}^{\mathbf{M}}$ is strictly below $\mathbf{K}$ in the Dodd-Jensen pre-well-ordering, yet $\mathbf{M}$ computes cardinal successors correctly for proper class many regular cardinals in $\mathbf{V}$. This example is essentially described in Steel [Ste96, §3]: It suffices to let $\mathbf{V}$ be the minimal iterable proper class extender model with one strong cardinal, say κ, and construct a linear iteration of length $\mathbf{On}$ by iteratively applying sufficiently large extenders on the images of the extender sequence of $\mathbf{V}$. Then let $\mathbf{M}$ be the initial segment of the resulting direct limit of ordinal length. We described this example here because it also shows that we cannot expect to prove universality of $\mathbf{K}^{\mathbf{M}}$ with respect to proper class extender models in $\mathbf{V}$ once $\mathbf{V}$ sees proper class inner models with strong cardinals. In the example just described, $\mathbf{V}$ sees a proper class inner model with a strong cardinal, namely itself, but $\mathbf{M}$ does not see any such inner model.

Corollary 7. *Assume* $\mathbf{M}$ *is a proper class inner model.*

(i) *If* δ *is a regular cardinal in* $\mathbf{V}$ *such that* $\delta^{+\mathbf{M}} = \delta^+$ *then the conclusions* (a) *and* (b) *in Theorem 5 hold.*
(ii) *If* $\delta^{+\mathbf{M}} = \delta^+$ *holds for proper class many cardinals* δ *that are regular in* $\mathbf{V}$ *then the conclusions in Corollary 6 hold.*

Proof. This follows from the observation that if δ is regular and $\delta^{+\mathbf{M}} = \delta^+$, then both S_δ and $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. $\square$

Notice that for $\delta = \omega_1$ in (i), the assumption that $\mathbf{M}$ computes the cardinal successor of δ correctly is equivalent to the requirement that $\omega_2^{\mathbf{M}} = \omega_2^{\mathbf{V}}$.

It is a folklore result that traces back at least to Hjorth's thesis (see the Claim in page 430 of Hjorth [Hjo95]) that not much can be concluded about the mice of an inner model $\mathbf{M}$ if only $\omega_1^{\mathbf{M}} = \omega_1$ is assumed. In particular, the following holds:

Fact 8. *If* $0^\sharp$ *exists, then there exists an inner model* $\mathbf{M}$ *which is a set forcing extension of* $\mathbf{L}$ *and computes* ω_1 *correctly. Thus* $0^\sharp \notin \mathbf{M}$.

Proof. Use the first ω_1 $\mathbf{L}$-indiscernibles,

$$(\iota_\alpha \mid \alpha \leq \omega_1^{\mathbf{V}}),$$

to guide an inductive construction of a $\mathsf{col}(\omega, < \omega_1^{\mathbf{V}})$-generic over $\mathbf{L}$ as follows:

- Only ω many dense sets need to be met to extend a $\mathsf{col}(\omega, < \iota_\alpha)$-generic G_α to a $\mathsf{col}(\omega, < \iota_{\alpha+1})$-generic $G_{\alpha+1}$ whenever α is countable.
- Since $\mathsf{col}(\omega, < \kappa)$ is κ-cc for κ regular, and the $\mathbf{L}$-indiscernibles form a club, it follows that $\bigcup_{\alpha<\kappa} G_\alpha$ is $\mathsf{col}(\omega, < \iota_\kappa)$-generic for $\kappa \leq \omega_1$ limit.

This gives the result. $\square$

Question 9. *Is there a class forcing extension of* $\mathbf{L}$ *such that no inner model with the same* ω_1 *is a set forcing extension of* $\mathbf{L}$*?*

On the other hand, as hinted by Lemma 3, assuming $\omega_2^{\mathbf{M}} = \omega_2$ is significantly different.

Corollary 10. *Assume that* $\mathbf{M}$ *is a proper class inner model, and that* $\mathcal{P}_{\omega_1}(\omega_2) \cap \mathbf{M}$ *is stationary. Let* r *be a sound mouse in* $\mathbf{V}$ *projecting to* ω *and below* $0^{¶}$*. Then* $r \in M$*.*

Proof. This follows from Theorem 5: Assume the conclusion fails, so $\mathbf{K}^{\mathbf{M}}$ exists and does not reach $0^{¶}$. Then, by Theorem 5.(b), $\mathbf{K}^{\mathbf{M}}||\omega_2$ is universal for countable mice in $\mathbf{V}$, and it follows from a standard comparison argument that such mice must be an initial segment of $\mathbf{K}^{\mathbf{M}}||\omega_2$. □

We reiterate that we made no anti-large cardinal assumptions on $\mathbf{V}$ in Corollary 10. Naturally, our results relativize so, for example, for any real $x \in \mathbf{M}$, Corollary 10 holds for countable x-mice below $x^{¶}$.

The proof of Theorem 5.(b) actually shows the following:

> If $0^{¶}$ exists in $\mathbf{V}$, then it is actually in $\mathbf{M}$.

Corollary 11. *Assume that* $\mathbf{M}$ *is a proper class inner model, and that* δ *is a regular cardinal in the sense of* $\mathbf{V}$*. Assume further that one of the following holds:*

(a) *In the sense of* $\mathbf{V}$*, there is no inner model with a Woodin cardinal,* $a^{\dagger}$ *exists for every real* a *in* $\mathbf{V}$*,* $\delta > \omega_1$*, and* S_δ *is stationary.*
(b) *In* $\mathbf{V}$*,* $0^{¶}$ *does not exist,* $\mathbb{R}^{\mathbf{V}}$ *is closed under sharps,* $\delta \geq \omega_1$*, and* $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ *is stationary .*

Then $\mathbf{M}$ *is* Σ^1_3*-correct.*

Proof. Let $A \subseteq \mathbb{R}$ be a nonempty Π^1_2-set. By Schindler [Schb] under assumption (a) and by Steel-Welch [SW98] or Schindler [Scha] under assumption (b), the true core model $\mathbf{K}$ of $\mathbf{V}$ is Σ^1_3-correct. That is, some $r \in A$ is an element of $\mathbf{K}$ and hence $r \in \mathbf{K}\,||\,\omega_1^{\mathbf{K}}$. By Theorem 5, the model $\mathbf{K}^{\mathbf{M}}\,||\,\delta$ iterates past $\mathbf{K}\,||\,\omega_1^{\mathbf{K}}$ if (a) holds or (b) holds and $\delta > \omega_1$, and $\mathbf{K}^{\mathbf{M}}\,||\,\omega_2$ iterates past $\mathbf{K}\,||\,\omega_1^{\mathbf{K}}$ if (b) holds and $\delta = \omega_1$. In either case we conclude that $r \in \mathbf{K}^{\mathbf{M}} \subseteq \mathbf{M}$, so $A \cap \mathbf{M}$ is nonempty. □

By Theorem 5, in Corollary 11.(b) we could equivalently assume that $0^{¶} \notin \mathbf{M}$, as such a sharp, if it exists in $\mathbf{V}$, must already be in $\mathbf{M}$ (as pointed out above.)

Corollary 12. *Assume that* $\mathbf{M}$ *is a proper class inner model, and that* δ *is a regular cardinal in* $\mathbf{V}$ *such that* $\delta^{+\mathbf{M}} = \delta^+$*. Suppose the assumptions in* (a) *or* (b) *of Corollary 11 hold where the requirement on the stationarity of* S_δ *or* $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ *is omitted. Then* $\mathbf{M}$ *is* Σ^1_3*-correct.*

Proof. As in the proof of Corollary 7. □

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2. Iterability and background core model theory

In this section we state facts from core model theory relevant to the proof of our main theorem, Theorem 5, and establish absoluteness of iterability for proper class extender models between two proper class inner models. The proof is standard, and we give it to make this paper self-contained. We also draw two useful corollaries, not be needed in our main argument.

Let us make some remarks concerning notation. As already mentioned in the introduction, we will be using Mitchell-Steel indexing of extenders in order to stay close enough to our primary reference Steel [Ste96]. If M is an extender model, E^M will denote its extender sequence, and E^M_β will denote the extender on the sequence E^M indexed by the ordinal β. Recall that if $\mathcal{T}$ is an iteration tree then $E^{\mathcal{T}}_\alpha$ is the extender coming from the α-th model $M^{\mathcal{T}}_\alpha$ used to form an ultrapower in $\mathcal{T}$, and $\delta(\mathcal{T})$ is the supremum of all ordinals $\rho(E^{\mathcal{T}}_\alpha)$; here $\rho(E^{\mathcal{T}}_\alpha)$ is the strict supremum of the generators of $E^{\mathcal{T}}_\alpha$. If $\mathcal{T}$ is of limit length then $\delta(\mathcal{T})$ is the supremum of all iteration indices in $\mathcal{T}$, that is, the supremum of all ordinals of the form $\mathsf{lh}(E^{\mathcal{T}}_\alpha)$. Next, $E(\mathcal{T})$ is the union of all extender sequences $E^{M^{\mathcal{T}}_\alpha} \restriction \rho(E^{\mathcal{T}}_\alpha)$ where $\alpha < \mathsf{lh}(\mathcal{T})$, and $M(\mathcal{T})$ is the premouse of height $\delta(\mathcal{T})$ whose extender sequence is $E(\mathcal{T})$. Recall also that if $\mathcal{T}$ is an iteration tree of limit length and b, c are two distinct cofinal well-founded branches through $\mathcal{T}$, then $\delta(\mathcal{T})$ is a Woodin cardinal in $M^{\mathcal{T}}_b \cap M^{\mathcal{T}}_c$ where $M^{\mathcal{T}}_b, M^{\mathcal{T}}_c$ are the models at the end of the corresponding branches; see for instance Mitchell-Steel [MS94]. Even if some of the branches b, c is ill-founded, $\delta(\mathcal{T})$ is a Woodin cardinal in $\mathsf{wfp}(M^{\mathcal{T}}_b) \cap \mathsf{wfp}(M^{\mathcal{T}}_c)$, that is, in the intersection of the well-founded parts of the two models. Granting

there is no proper class inner model with a Woodin cardinal, any iterable proper class extender model has precisely one iteration strategy called the **uniqueness** strategy; this is the strategy that picks the unique cofinal well-founded branch at each limit step of the iteration.

The following lemma is a slight variation of Steel [Ste96, Lemma 5.12], and may be viewed as a generalization thereof, as we allow $\mathbf{V}$ be any outer extension of $\mathbf{M}$, and not merely a generic extension. The proof thus requires an additional appeal to Shoenfield absoluteness, which somewhat changes not only the structure of the proof but the statemtent of the result itself. We give the proof below; one notable difference is that the proof of Steel [Ste96, Lemma 5.12] uses, in an important way, the assumption that every set in the ground model has a sharp. The existence of sharps is, of course, an immediate consequence of the initial setting where Ω is a measurable cardinal and our universe of interest is the rank-initial segment of height Ω. However, this assumption may be false if we want to run the proof of Theorem 5 in ZFC alone. There is one more difference between the two proofs which makes our lemma in a sense "less general" than Steel [Ste96, Lemma 5.12], as we are making the assumption that there is no inner model with a Woodin cardinal in the sense of $\mathbf{V}$. The proof of Steel [Ste96, Lemma 5.12] only requires that no Woodin cardinals exist in the extender model we want to iterate. We do not know if the asumption on the non-existence of an inner model with a Woodin cardinal in the sense of $\mathbf{V}$ can be weakended by replacing $\mathbf{V}$ with the model we intend to iterate or with our inner model $\mathbf{M}$, even in the case where we allow our universe to be closed under sharps.

Lemma 13. *Assume that* $\mathbf{M}$ *is a proper class inner model, and that in* $\mathbf{V}$ *there is no proper class inner model with a Woodin cardinal. In* $\mathbf{M}$*, let* R *be a proper class extender model. The following are equivalent:*

(a) *In* $\mathbf{M}$*, the model* R *is normally iterable via the uniqueness strategy.*
(b) *In* $\mathbf{V}$*, the model* R *is normally iterable via the uniqueness strategy.*

Here "iterable" means iterable with respect to trees of set-sized length.

Proof. We give a proof of the implication (a) $\Rightarrow$ (b). The converse is proved in a similar way, and the argument is much simpler. Toward a contradiction, assume that in $\mathbf{V}$ there is a normal iteration tree $\mathcal{T}$ on R witnessing that (a) $\Rightarrow$ (b) is false. We focus on the essential case where the length of $\mathcal{T}$ is a limit ordinal. Thus, in $\mathbf{V}$ the tree $\mathcal{T}$ does not have a unique cofinal well-founded branch. Using a reflection argument, we can find a regular cardinal $\alpha > \delta(\mathcal{T})^+$ such that $\mathcal{T}$, when viewed as a tree on $R' = R\,||\,\alpha$, does not have a unique cofinal well-founded branch in $\mathbf{V}$. Notice that the transitive closure of $\mathcal{T}$ is then of size α. Let $\theta > \alpha$ be a limit cardinal with large cofinality, and let T be the lightface theory of $H_\theta^{\mathbf{V}}$ in the language of ZFC. Let $\varphi(x, y, z)$ be the conjunction of the following statements:

- x is a transitive model of T.
- $z, y \in x$, and z is a regular cardinal in x.

- There is a normal iteration tree $\mathcal{U} \in x$ on the premouse y of limit length such that $\mathsf{card}(\mathsf{trcl}(\mathcal{U}))^x = z$, $\delta(\mathcal{U})^{+x} < z$, and in x, the tree $\mathcal{U}$ does not have a unique cofinal well-founded branch.
- There is no $a \in x$ such that a is a bounded subset of z and $J_z[a]$ is a model with a Woodin cardinal.

Here $\mathsf{trcl}(v)$ denotes the transitive collapse of v.

Let $\beta \geq \mathsf{card}(H_\theta)$, and let G be a filter generic for $\mathsf{col}(\omega, \beta)$ over $\mathbf{V}$. Immediately from the properties of $\mathcal{T}$ and from our assumption that no inner model with a Woodin cardinal exists in the sense of $\mathbf{V}$ we conclude that $\varphi(H_\theta^{\mathbf{V}}, R', \alpha)$ holds in $\mathbf{V}[G]$. As $H_\theta^{\mathbf{V}}$ is countable in $\mathbf{V}[G]$ and R', α are countable in $\mathbf{M}[G]$, it follows by Shoenfield absoluteness that in $\mathbf{M}[G]$ there is some H such that $\mathbf{M}[G] \models \varphi(H, R', \alpha)$.

Pick a $\mathsf{col}(\omega, \beta)$-name $\dot{H} \in \mathbf{M}$ and some regular θ' such that $H = \dot{H}_G$ and $\beta, \dot{H}, R' \in H_{\theta'}^{\mathbf{M}}$. Finally, pick a condition $p \in \mathsf{col}(\omega, \beta)$ such that p forces $\varphi(\dot{H}, \check{R}', \check{\alpha})$ over $H_{\theta'}^{\mathbf{M}}$. From now on, work in $\mathbf{M}$. Let X be a countable elementary substructure of $H_{\theta'}^{\mathbf{M}}$ such that $\dot{H}, R', \beta, p \in X$, and let

$$\sigma : \tilde{H} \to H_{\theta'}^{\mathbf{M}}$$

be the inverse of the Mostowski collapsing isomorphism that arises from collapsing X. Let $\dot{\bar{H}}, \bar{R}, \bar{\alpha}, \bar{\beta}, \bar{p}$ be the preimages of $\dot{H}, R', \alpha, \beta, p$ under σ. Pick a filter g extending $\bar{p}$ that is generic for $\mathsf{col}(\omega, \bar{\beta})$ over $\tilde{H}$, and let $\bar{H} = \dot{\bar{H}}_g$. Then $\bar{\alpha}$ is a regular cardinal in $\bar{H}$, and there is a normal iteration tree $\bar{\mathcal{U}} \in \bar{H}$ on $\bar{R}$ of limit length such that $\mathsf{card}(\mathsf{trcl}(\bar{\mathcal{U}}))^{\bar{H}} = \bar{\alpha}$, $\delta(\bar{\mathcal{U}})^{+\bar{H}} < \bar{\alpha}$ and, in $\bar{H}$, the tree $\bar{\mathcal{U}}$ does not have a unique cofinal well-founded branch.

We next observe that for every limit ordinal $\lambda^* < \bar{\lambda}$, the tree $\bar{\mathcal{U}}$ has at most one cofinal well-founded branch in $\mathbf{M}$: If $b \neq c$ were two distinct cofinal well-founded branches through $\bar{\mathcal{U}} \restriction \lambda^*$, then $\delta(\bar{\mathcal{U}} \restriction \lambda^*)$ would be Woodin in $M_b^{\bar{\mathcal{U}}} \cap M_c^{\bar{\mathcal{U}}}$. If $\nu > \delta(\bar{\mathcal{U}} \restriction \lambda^*)$ is least among all ordinals indexing extenders in $M_b^{\bar{\mathcal{U}}}$ or $M_c^{\bar{\mathcal{U}}}$ (note such a ν always exists as $\lambda^* < \bar{\lambda}$), assume without loss of generality that ν indexes an extender in $M_b^{\bar{\mathcal{U}}}$. Then $\delta(\bar{\mathcal{U}} \restriction \lambda^*)$ is Woodin in $M_b^{\bar{\mathcal{U}}} \,||\, \nu$ and, using the elementarity of the iteration maps, we conclude that there are $\delta' < \nu'$ such that ν' indexes an extender in $\bar{R}$, and λ' is Woodin in $\bar{R} \,||\, \nu'$. Since $\bar{R} \in \tilde{H}$, we may apply σ to $\bar{R}$ and conclude that $\sigma(\nu')$ indexes an extender in R' and $\sigma(\delta')$ is Woodin in $R' \,||\, \sigma(\nu') = R \,||\, \sigma(\nu')$. Since we assume R is iterable in the sense of $\mathbf{M}$, iterating the extender on the R-sequence with index ν through the ordinals will produce a proper class inner model with a Woodin cardinal, a contradiction.

The same argument for $\lambda^* = \bar{\lambda}$ shows that in $\mathbf{M}$, if $b \neq c$ are two cofinal well-founded branches through $\bar{\mathcal{U}}$ then no $\nu > \delta(\bar{\mathcal{U}})$ indexes an extender in $M_b^{\bar{\mathcal{U}}}$ or $M_c^{\bar{\mathcal{U}}}$. Now at most one of the branches b, c involves a truncation; otherwise the standard argument can be used to produce two compatible extenders applied in $\bar{\mathcal{U}}$, and thereby obtain a contradiction. If one of the branches involves a truncation, let it be without loss of generality c; then

$M_b^{\bar{\mathcal{U}}} \trianglelefteq M_c^{\bar{\mathcal{U}}}$. If neither of the branches involves a truncation, let b be such that $M_b^{\bar{\mathcal{U}}} \trianglelefteq M_c^{\bar{\mathcal{U}}}$. In either case, $\bar{\lambda}$ is Woodin in $M_b^{\bar{\mathcal{U}}}$ and there is an iteration map $\pi_b : \bar{R} \to M_b^{\bar{\mathcal{U}}}$. By the elementarity of π_b, there is a Woodin cardinal in $\bar{R}$. And by the elementarity of σ, there is a Woodin cardinal in R'. Since $R' = R\,||\,\alpha$ and α is a cardinal, R is a proper class model with a Woodin cardinal, a contradiction. In particular, this shows that $\bar{\mathcal{U}}$ has no cofinal well-founded branch in $\bar{H}$. (Because our initial hypothesis stipulates that such a branch cannot be unique.)

Since $\sigma \restriction \bar{R} : \bar{R} \to R'$ is elementary and $\bar{\mathcal{U}}$ picks unique cofinal well-founded branches, we can copy $\bar{\mathcal{U}}$ onto an R'-based iteration tree $\mathcal{U}'$ via $\sigma \restriction \bar{R}$. Let b be a cofinal well-founded branch through $\mathcal{U}'$; such a branch exists by our assumption on the iterability of R in $\mathbf{M}$. By the properties of the copying construction, b is also a cofinal well-founded branch through $\bar{\mathcal{U}}$. Now assume $\gamma \geq \bar{\alpha}$ is an ordinal in $\bar{H}$, and $h \in \mathbf{M}$ is a filter on $\mathsf{col}(\omega,\gamma)$ that is generic over $\bar{H}$. By our initial setting, $\mathsf{card}(\mathsf{trcl}(\mathcal{U}))^{\bar{H}} = \bar{\alpha} \leq \gamma$; hence both γ and the transitive closure of $\bar{\mathcal{U}}$ are countable in $\bar{H}[h]$. Since the statement "There is a cofinal branch x through $\bar{\mathcal{U}}$ such that the ordinals of $M_x^{\bar{\mathcal{U}}}$ have an initial segment isomorphic to γ" is Σ^1_1 in the codes, if $\bar{\mathcal{U}}$ has such a branch, then some such branch c exists in $\bar{H}[h]$. If $M_c^{\bar{\mathcal{U}}}$ is well-founded, then by the previous paragraph, c is unique, and by the homogeneity of $\mathsf{col}(\omega,\gamma)$, such a branch must be in $\bar{H}$, which contradicts the last conclusion of the previous paragraph. Hence $M_c^{\bar{\mathcal{U}}}$ must be ill-founded. If $\mathbf{On} \cap M_b^{\bar{\mathcal{U}}} < \mathbf{On} \cap \bar{H}$ put $\gamma = \mathbf{On} \cap M_b^{\mathcal{U}}$, and apply the above argument to the Σ^1_1-statement "There is a cofinal branch x through $\bar{\mathcal{U}}$ such that the ordinals of $M_x^{\bar{\mathcal{U}}}$ are isomorphic to γ". We obtain a cofinal well-founded branch through $\bar{\mathcal{U}}$ in $\bar{H}[h]$, a contradiction. (Here we are using the fact that there are unboundedly many cardinals in $\bar{H}$.) It follows that $M_b^{\bar{\mathcal{U}}}$ has height at least that of $\bar{H}$.

Let $\tau = \delta(\bar{\mathcal{U}})^{+\mathbf{L}[E(\bar{\mathcal{U}})]}$. Set $\gamma = \alpha$, and let $h \in \mathbf{M}$ be generic for $\mathsf{col}(\omega,\gamma)$ over $\bar{H}$. The arguments from the previous paragraph imply that there is a branch $c \in \bar{H}[h]$ that has an initial segment isomorphic to γ. As follows from the previous paragraph, such a branch is ill-founded. Let γ' be a cardinal in $\bar{H}$ larger than the ordinal of the well-founded part of $M_c^{\bar{\mathcal{U}}}$, and let h' be a filter generic for $\mathsf{col}(\omega,\gamma')$ over $\bar{H}$. As $\mathbf{On} \cap M_b^{\bar{\mathcal{U}}} > \gamma'$, as before we conclude that there is a branch c' in $\bar{H}[h']$ with an initial segment isomorphic to γ'. (Here we again make use of the fact that cardinals of $\bar{H}$ are unbounded in $\bar{H}$.)

Since the well-founded part of $M_{c'}^{\bar{\mathcal{U}}}$ is strictly longer than the well-founded part of $M_c^{\bar{\mathcal{U}}}$, we conclude that $c \neq c'$. And since both γ, γ' are larger than $\alpha > \tau$, the ordinal $\delta(\bar{\mathcal{U}})$ is a Woodin cardinal in $J_\alpha[E(\bar{\mathcal{U}})]$. Since $\bar{\mathcal{U}} \in \bar{H}$, so is $J_\alpha[E(\bar{\mathcal{U}})]$, but this contradics the last clause in φ which holds in $\bar{H}$ by construction. $\square$

Notice that the only place where the assumption on the non-existence of a proper class inner model with a Woodin cardinal in the sense of $\mathbf{V}$ was used was at the end of the very last paragraph in the proof of Lemma 13. All other appeals to a smallness hypotheses used merely the assumption that R is 1-small.

As a by-product, we note the following interesting consequence of Lemma 13.

Corollary 14. *Assume that in $\mathbf{V}$, there is no proper class inner model with a Woodin cardinal. Let R, R' be proper class extender models and $\sigma : R \to R'$ be an elementary map. Then R is normally iterable if and only if R' is normally iterable. Here "normally iterable" means that there is a uniqueness normal iteration strategy defined on all set-sized normal iteration trees.*

Proof. Assume R is iterable in $\mathbf{V}$. Applying Lemma 13 to the situation where $\mathbf{M} = R$, we conclude that R is internally normally iterable, that is, R satisfies the statement "R is normally iterable". By the elementarity of the map σ, the model R' is internally normally iterable. Applying Lemma 13 again, this time with $\mathbf{M} = R'$, we conclude that R' is normally iterable in the sense of $\mathbf{V}$. □

Recall that an extender model R is **fully iterable** if and only if R is iterable with respect to linear compositions of arbitrary (hence even class-sized) normal iteration trees. When working in a universe below the measurable cardinal Ω, normal iterability with respect to set-sized trees automatically guarantees normal iterability with respect to class-sized trees. This, in turn, can be used to prove the full iterability of R'.

Working in the universe below Ω, fix a stationary class of ordinals that will serve as a reference class for the notion of *thickness*. In our applications we will assume for simplicity that this class consists of inaccessibles, so we let this class be the class A_0 as defined at the end of Steel [Ste96, §1]. Then $\mathbf{K}^c$ is A_0-thick.

Corollary 15. *Assume that in $\mathbf{V}$, there is no proper class inner model with a Woodin cardinal. Let R be a proper class extender model that is fully iterable and A_0-thick, and let F be an R-extender such that $\mathsf{Ult}(R, F)$ is well-founded. Denote the transitive collapse of $\mathsf{Ult}(R, F)$ by R'. Then R' is fully iterable.*

Proof. By Corollary 14 and the discussion immediately following, R' is normally iterable with respect to class-sized normal iteration trees. It follows that there is an extender model R^* that is a common normal iterate of both R and R'. Since R is thick, the corresponding normal iteration tree on R' does not involve any truncation on its main branch, and there is an iteration map $\pi : R' \to R^*$ that is fully elementary. Now R^* is fully iterable, as it is an iterate of R. The existence of π then guarantees that R' is fully iterable as well. □

Finally we will need a well-known lemma on the absorbtion of extenders by the core model.

Lemma 16. *Assume that there is no proper class model with a Woodin cardinal. Let F be a $\mathbf{K}$-extender with support λ and let W be a proper class extender model witnessing the A_0-soundness of $\mathbf{K}\,||\,\delta$ where δ is a cardinal larger than λ. Assume further that the following hold.*

(a) *The phalanx $\langle W, W', \lambda\rangle$ is normally iterable where $W' = \mathsf{Ult}(W, F)$.*
(b) *$E^W \restriction \nu = E^{W'} \restriction \nu$ where $\nu = \lambda^{+W'}$.*

Then $F \in W$.

Proof. See, for example, Steel [Ste96, Theorem 8.6] □

In general, if F is a (κ, λ) extender that satisfies condition (b) above, we say that F **coheres** to W.

3. Universality

3.1. **Outline.** Throughout this section we will work with a fixed proper class inner model $\mathbf{M}$ and a cardinal δ which is regular and uncountable in $\mathbf{V}$. In order to guarantee that the core model theory is applicable, we will additionally assume that in $\mathbf{V}$ there is no proper class inner model with a Woodin cardinal.

Our argument is based on a combination of ideas coming from the proof of universality of $\mathbf{K}^c$ described in Jensen [Jenb] and, in the simpler form reformulated for measures of order zero, also in Zeman [Zem02, §6.4], further from the proof of universality of $\mathbf{K}^c$ in Mitchell-Schindler [MS04], and finally from the Mitchell-Schimmerling-Steel proof of universality of $\mathbf{K}\,||\,\delta$, see Schimmerling-Steel [SS99]. But of course, since we work under singificantly restricted circumstances, we need to do some amount of extra work.

As it is usual in arguments of this kind, we will replace $\mathbf{K}^{\mathbf{M}}$, the true core model in $\mathbf{M}$, with a proper class fully iterable extender model W that in $\mathbf{M}$ witnesses the A_0-soundness of $\mathbf{K}^{\mathbf{M}}\,||\,\delta'$ for a sufficiently large $\delta' > \delta$. This means that, working in $\mathbf{M}$, the model W extends $\mathbf{K}^{\mathbf{M}}\,||\,\delta'$, is A_0-thick, and $\delta' \subseteq H_1^W(\Gamma)$ whenever Γ is A_0-thick in W. (See Steel [Ste96, §3] for details.) In our case, taking $\delta' = \delta^+$ will suffice. We recall that we are using Mitchell-Steel indexing of extenders, as this setting is convenient when referring to Steel [Ste96].

Our general strategy is the following:

(A) Assuming that in $\mathbf{V}$, there is an iterable premouse of size less than δ that iterates past $W\,||\,\delta$, we construct a W-extender with critical point κ and support λ such that $F \in \mathbf{M}$ and coheres to W. (This does not require that $\mathsf{Ult}(W, F)$ is well-founded.)
(B) We prove that $\mathsf{Ult}(W, F)$ is well-founded and, letting W' be its transitive collapse, prove that the phalanx (W, W', λ) is normally iterable in the sense of $\mathbf{M}$. We then apply Lemma 16 inside $\mathbf{M}$.

3.2. **Part (A).** We begin with (A). The construction in (A) is the same for both parts of Theorem 5; as mentioned above, we will use the Mitchell-Schindler approach to the proof of universality. Assume that

$$S \subseteq \mathcal{P}(\delta^+) \cap \mathbf{M} \tag{2}$$

is stationary in the sense of $\mathbf{V}$.

By Lemma 13, the model W is normally iterable via the uniqueness strategy in the sense of $\mathbf{V}$, so in particular it is coiterable with any iterable premouse in $\mathbf{V}$. Assume N is an iterable premouse in $\mathbf{V}$ of size less than δ with an iteration strategy Σ that iterates past $\mathbf{K}^{\mathbf{M}} \,||\, \delta$. This means that there is a pair $(\mathcal{T},\mathcal{U})$ of normal iteration trees arising in the coiteration of W against N, where we use the uniqueness strategy for W and Σ for N, such that the length of $\mathcal{U}$ is $\delta+1$. We view $\mathcal{T},\mathcal{U}$ as unpadded trees, so the length of $\mathcal{T}$ is $\leq \delta+1$. Also, we view $\mathcal{T}$ as an iteration tree on $W \,||\, \delta^+$.

Fix the following notation:

- $\mathcal{T}$, resp. $\mathcal{U}$, is the iteration tree of length $\delta+1$ on the W-side, resp. the N-side, and W', resp. N' is the last model on the respective side. So $\mathcal{T}$ is according to the uniqueness strategy and $\mathcal{U}$ is according to Σ. We treat the trees $\mathcal{T},\mathcal{U}$ as if they were not padded.
- The notation $E^{\mathcal{T}}_i, \kappa^{\mathcal{T}}_i, \pi^{\mathcal{T}}_{ij}$ is self-explanatory, and $\nu^{\mathcal{T}}_i$ is the iteration index associated with the ν_i-th extender on $\mathcal{T}$, that is, $\nu^{\mathcal{T}}_i = \mathsf{lh}(E^{\mathcal{T}}_i)$. We of course fix the corresponding notation for $\mathcal{U}$.
- $W_i = M^{\mathcal{T}}_i$ and $N_i = M^{\mathcal{U}}_i$.
- T, resp. U, is the tree structure associated with the iteration tree $\mathcal{T}$, resp. $\mathcal{U}$, and $<_T$, $<_U$ are the respective tree orderings.
- $b^{\mathcal{T}}$, resp. $b^{\mathcal{U}}$, is the branch of $\mathcal{T}$, resp. $\mathcal{U}$, that contains δ. We will also refer to these branches as main branches.
- If b is a cofinal branch through $\mathcal{T}$, we write $\pi^{\mathcal{T}}_b$ to denote the iteration map along b. If $i \in b$, we write $\pi^{\mathcal{T}}_{i,b}$ to denote the iteration map with domain $M^{\mathcal{T}}_i$ into the direct limit $M^{\mathcal{T}}_b$, so $\pi^{\mathcal{T}} = \pi^{\mathcal{T}}_{0,b}$.
- For any $i < \omega_1$ we write $\xi^{\mathcal{T}}(i)$, resp. $\xi^{\mathcal{U}}(i)$, to denote the immediate $<_T$-, resp. $<_U$-, predecessor of $i+1$.

Lemma 17. *Under the above setting, the following hold:*

(a) *There is no truncation point on $b^{\mathcal{T}}$.*
(b) *For every $\alpha < \delta$ there is $i+1 \in b^{\mathcal{T}}$ such that $\pi^{\mathcal{T}}_{0,\xi(i)}(\alpha) < \kappa^{\mathcal{T}}_i = \mathsf{cr}(\pi^{\mathcal{T}}_{\xi(i),b^{\mathcal{T}}})$. Equivalently, $\pi^{\mathcal{T}}_{b^{\mathcal{T}}}(\delta) = \delta$.*

Proof. Both clauses follow by the standard argument that proves the termination of the comparison process. □

Using a pressing down argument, we obtain a club $C^* \subseteq b^{\mathcal{U}}$ which is a thread through the set of critical points. Precisely, $\pi^{\mathcal{U}}_{\xi,\xi'}(\kappa^{\mathcal{U}}_i) = \kappa^{\mathcal{U}}_j$ whenever $\xi < \xi'$ are elements of C^* such that $\xi = \xi^{\mathcal{U}}(i)$ and $\xi' = \xi^{\mathcal{U}}(j)$. Of course

$\min(C^*) \geq \xi^{\mathcal{U}}(i^{\mathcal{U}})$ where $i^{\mathcal{U}}+1$ is the last truncation point on $b^{\mathcal{U}}$. We will additionally assume that elements of C^* satisfy the following:

(i) $\xi = \kappa_i^{\mathcal{U}}$ whenever $\xi \in C^*$ and i is such that $\xi = \xi^{\mathcal{U}}(i)$, hence $\xi = \mathsf{cr}(\pi^{\mathcal{U}}_{\xi,\xi'})$ for all $\xi' \in b^{\mathcal{U}} - (\xi+1)$.
(ii) $\pi^{\mathcal{T}}_{0,\xi}(\xi) = \xi$.
(iii) ξ is a limit point of the critical points on $b^{\mathcal{T}}$.

That these requirements may be imposed without loss of generality follows from the fact that they are true on a club. For (i) this is obvious, and for (ii) and (iii) we appeal to the proof of Lemma 17. Notice also that each element of C^* is an inaccessible cardinal in N', and since N' agrees with W' below δ, also in W'.

Lemma 18. *Given any $\xi \in C^*$, let $\tau_\xi = \xi^{+W}$ and $\tau'_\xi = \xi^{+W_\xi}$. Then for every $\xi \in C^*$ the following is true:*

(a) *ξ is inaccessible in W_i for all $i \in b^{\mathcal{T}}$.*
(b) *$\tau'_\xi = \xi^{+N_\xi} = \xi^{+N'}$.*

Proof. Let $\xi \in C^*$ and i be such that $\xi = \xi^{\mathcal{T}}(i)$. Since the critical points on $b^{\mathcal{T}}$ are cofinal in ξ, necessarily $\kappa_i^{\mathcal{T}} \geq \xi$. The critical point of the map $\pi^{\mathcal{T}}_{i+1,\delta}$ is above ξ, so ξ is an inaccessible cardinal in W_{i+1}. Since $W_{i+1} = \mathsf{Ult}(W_\xi, E_i^{\mathcal{T}})$ and $E_i^{\mathcal{T}}$ is an extender with critical point at least ξ, the ordinal ξ remains inaccessible in W_ξ. Since $\pi^{\mathcal{T}}_{0,\xi}(\xi) = \xi$, the ordinal ξ remains inaccessible in all models W_i for $i <_T \xi$. This proves (a). Regarding (b), we have $\mathcal{P}(\xi) \cap W_{\xi^{\mathcal{T}}(i)} = \mathcal{P}(\xi) \cap W_{i+1}$, by our choice of ξ and general properties of iterations trees. Since $\mathsf{cr}(\pi^{\mathcal{T}}_{i+1}) > \xi$, we conclude $\tau'_\xi = \xi^{+W'} = \xi^{+N'} = \xi^{N_\xi}$, where the last equality follows by the same considerations for $\mathcal{U}$ as have been just done for $\mathcal{T}$. □

The initial setting recorded above is more or less the same as in Jensen's proof of universality of $\mathbf{K}^c$ in Jensen [Jenb]. Although the next two lemmata follow the general scenario of his proof, the arguments we use are significantly more local, and we need to do this for two reasons. First, we work in $\mathbf{V}$ but on the other hand, the model whose universality we want to establish is in $\mathbf{M}$, which restricts our freedom in choosing the structures to work with. Second, we work below a regular cardinal δ given in advance which – unlike the situation in Jensen's proof – we would like to be as small as possible, and therefore it need not have any reasonable closure properties. We stress that we are not making any assumptions on cardinal arithmetic. These two lemmata comprise one of the key observations in our argument.

Lemma 19. *Let $\xi \in \lim(C^*)$ and $\xi < \zeta < \tau_\xi$. Then there is a $\xi^* \in C^*$ such that $\xi^* < \xi$ and, for all $\bar{\xi} \in C^*$ with $\xi^* \leq \bar{\xi} < \xi$, we have*

$$(\pi^{\mathcal{U}}_{\bar{\xi},\xi})^{-1} \circ \pi^{\mathcal{T}}_{0,\xi}(\zeta) \in \operatorname{rng}(\pi^{\mathcal{T}}_{0,\bar{\xi}}).$$

Proof. Since $\xi < \zeta < \tau_\xi$, there is a set $a \in W$ such that $a \subseteq \xi \times \xi$ and a is a well-ordering of ξ of order-type of ζ. Let ξ^* be the least element ξ' of C^* such that $\pi^{\mathcal{T}}_{0,\xi}(a) \in \mathrm{rng}(\pi^{\mathcal{U}}_{\xi',\xi})$. Then for $\bar\xi \in [\xi^*, \xi) \cap C^*$ we have

$$\pi^{\mathcal{T}}_{0,\xi}(a) = \pi^{\mathcal{U}}_{\bar\xi,\xi}(\pi^{\mathcal{T}}_{0,\xi}(a) \cap \bar\xi) = \pi^{\mathcal{U}}_{\bar\xi,\xi} \circ \pi^{\mathcal{T}}_{0,\bar\xi}(a \cap \bar\xi).$$

Here the first equality follows from the fact that $\bar\xi = \mathsf{cr}(\pi^N_{\bar\xi,\xi})$ and $\pi^N_{\bar\xi,\xi}(\bar\xi) = \xi$. To see the second equality notice that $\mathsf{cr}(\pi^M_{\bar\xi,\xi}) \geq \bar\xi$, so $\pi^{\mathcal{T}}_{0,\xi}(a) \cap \xi = \pi^{\mathcal{T}}_{0,\bar\xi}(a)$. By the elementarity of all relevant maps, $a \cap \bar\xi$ is a well-ordering on $\bar\xi$. Letting $\bar\zeta$ be the order-type of $a \cap \bar\xi$, we conclude that $(\pi^{\mathcal{U}}_{\bar\xi,\xi})^{-1} \circ \pi^{\mathcal{T}}_{0,\xi}(\zeta) = \pi^{\mathcal{T}}_{0,\bar\xi}(\bar\zeta)$. $\square$

Given a map f and a set x, we will write $f[x]$ to denote the pointwise image of x under f.

Lemma 20. *There is a $\xi^* \in C^*$ such that for every $\xi, \xi' \in C^*$ with $\xi^* \leq \xi < \xi'$ we have $\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}[W \,||\, \tau_\xi] \subseteq \pi^{\mathcal{T}}_{0,\xi'}[W \,||\, \tau_{\xi'}]$. Moreover, the set on the left is a cofinal subset of the set on the right.*

Proof. To prove the inclusion, it suffices to show that $\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}[\tau_\xi] \subseteq \pi^{\mathcal{T}}_{0,\xi'}[\tau_{\xi'}]$ for all $\xi < \xi'$ on a tail-end of C^*. The first step toward the proof of this inclusion is the following claim. Let $\gamma = \min(C^*)$.

Claim 21. *Let $\gamma < \beta < \tau'_\gamma$. There is a $\xi_\beta \in C^*$ and a sequence*

$$\langle \zeta_\xi \mid \xi \in C^* - \xi_\beta \rangle$$

such that the following holds:

(a) *$\xi < \zeta_\xi < \tau_\xi$ and $\pi^{\mathcal{T}}_{0,\xi}(\zeta_\xi) \geq \pi^{\mathcal{U}}_{\gamma,\xi}(\beta)$ whenever $\xi \in C^* - \xi_\beta$, and*
(b) *$\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(\zeta_\xi) = \pi^{\mathcal{T}}_{0,\xi'}(\zeta_{\xi'})$ whenever $\xi < \xi'$ are in $C^* - \xi_\beta$.*

Proof. Pick a sequence $\langle \bar\zeta_\xi \mid \xi \in \lim(C^*) \rangle$ such that $\xi < \bar\zeta_\xi < \tau_\xi$ and $\pi^{\mathcal{T}}_{0,\delta}(\bar\zeta_\xi) \geq \pi^{\mathcal{U}}_{\gamma,\xi}(\beta)$ for all $\xi \in \lim(C^*)$. By Lemma 19 we obtain a regressive function g defined on $\lim(C^*)$ such that $(\pi^{\mathcal{U}}_{\bar\xi,\xi})^{-1} \circ \pi^{\mathcal{T}}_{0,\xi}(\bar\zeta_\xi) \in \mathrm{rng}(\pi^{\mathcal{T}}_{0,\bar\xi})$ whenever $\xi \in \lim(C^*)$ and $\bar\xi \in [g(\xi), \xi) \cap C^*$. Using Fodor's lemma, we find a $\xi_\beta \in C^*$ and a stationary set $A'_\beta \subseteq \lim(C^*)$ such that $g(\xi) = \xi_\beta$ for all $\xi \in A'_\beta$. Then, using the pigeonhole principle (recall that $\mathsf{card}(N_{\xi_\beta}) < \delta$), we find an ordinal ζ'_β such that $\xi_\beta < \zeta'_\beta < \tau'_{\xi_\beta}$ (see Lemma 18 for the definition of τ'_η), and a stationary set $A_\beta \subseteq A'_\beta$ such that $\pi^{\mathcal{U}}_{\xi_\beta,\xi}(\zeta'_\beta) = \pi^{\mathcal{T}}_{0,\xi}(\bar\zeta_\xi)$ for all $\xi \in A_\beta$. For $\xi \in C^* - \xi_\beta$ then let

$$\zeta_\xi = (\pi^{\mathcal{T}}_{0,\xi})^{-1} \circ \pi^{\mathcal{U}}_{\xi_\beta,\xi}(\zeta'_\beta).$$

Notice that ζ_ξ is defined for every $\xi \in C^* - \xi_\beta$: Just pick any $\xi' \in A_\beta$ such that $\xi' \geq \xi$; then

$$\pi^{\mathcal{U}}_{\xi_\beta,\xi}(\zeta'_\beta) = (\pi^{\mathcal{U}}_{\xi,\xi'})^{-1} \circ \pi^{\mathcal{T}}_{0,\xi'}(\bar\zeta_\beta)$$

and the right side is in the range of $\pi^{\mathcal{T}}_{0,\xi}$ by Lemma 19. (Notice that for $\xi \in A_\beta$ we have $\zeta_\xi = \bar\zeta_\xi$.) Clause (a) of the Claim then follows easily from

our choice of ordinals $\bar{\zeta}_\xi$. Now if $\xi < \xi'$ are in $C^* - \xi_\beta$, then

$$\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(\zeta_\xi) = \pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{U}}_{\xi_\beta,\xi}(\zeta'_\beta) = \pi^{\mathcal{U}}_{\xi_\beta,\xi'}(\zeta'_\beta) = \pi^{\mathcal{T}}_{0,\xi'}(\zeta_{\xi'}),$$

which verifies (b) of the Claim. □

We now complete the proof of Lemma 20. Let $\varepsilon = \mathsf{cf}^{\mathbf{V}}(\tau'_\gamma)$. Pick a sequence $\langle \beta_i \mid i < \varepsilon \rangle$ that is increasing and cofinal in τ'_γ. Appealing to the above Claim, for each $i < \varepsilon$ there is $\xi^i = \xi_{\beta_i}$ and a sequence $\langle \zeta^i_\xi \mid \xi \in C^* - \xi^i \rangle$ such that (a) and (b) of the Claim hold with ξ^i, ζ^i_ξ in place of ξ_β, ζ_ξ. Let $\xi^* \in C^*$ be an upper bound on all ξ^i; such a ξ^* exists as $\varepsilon < \delta$. Then (a) and (b) of the above Claim hold with ξ^*, ζ^i_ξ in place of ξ_β, ζ_ξ whenever $i < \varepsilon$. Also, since each $\pi^{\mathcal{T}}_{0,\xi}$ maps τ_ξ cofinally into τ'_ξ, and $\pi^{\mathcal{U}}_{\gamma,\xi}$ maps τ'_γ cofinally into τ'_ξ, each sequence $\langle \zeta^i_\xi \mid i < \varepsilon \rangle$ is cofinal in τ_ξ. It follows that for any $\xi < \xi'$ in $C^* - \xi^*$, the assignment $\zeta^i_\xi \mapsto \zeta^i_{\xi'}$ maps a cofinal subset of τ_ξ cofinally into $\tau_{\xi'}$.

Fix $\xi < \xi'$ in $C^* - \xi^*$ and an $i < \gamma$. Let $f, f' \in W$ be such that f is the $<_{E^W}$-least surjection of ξ onto $M \,||\, \zeta^i_\xi$ and f' is the $<_{E^W}$-least surjection of ξ' onto $W \,||\, \zeta^i_{\xi'}$; recall that E^W is the extender sequence of W. Then $f \in W \,||\, \tau_\xi$ and $f' \in W \,||\, \tau_{\xi'}$. Appealing to (b) in the Claim and to the elementarity of the maps $\pi^{\mathcal{T}}_{0,\xi}$ and $\pi^{\mathcal{U}}_{\xi,\xi'}$ we conclude that $\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(f) = \pi^{\mathcal{T}}_{0,\xi'}(f')$. Given $x \in W \,||\, \zeta^i_\xi$, let $\eta < \xi$ be such that $x = f(\eta)$. Then

$$\begin{aligned}
\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(x) &= \pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(f(\eta)) = \pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(f)\left(\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}(\eta)\right) \\
&= \pi^{\mathcal{T}}_{0,\xi'}(f')(\pi^{\mathcal{T}}_{0,\xi}(\eta)) = \pi^{\mathcal{T}}_{0,\xi'}(f')\left((\pi^{\mathcal{T}}_{0,\xi'}(\eta)\right) = \pi^{\mathcal{T}}_{0,\xi'}(f'(\xi)) \\
&= \pi^{\mathcal{T}}_{0,\xi'}(x).
\end{aligned}$$

Here the first two equalities on the middle line follow from the fact that $\mathsf{cr}(\pi^{\mathcal{U}}_{\xi,\xi'}) = \xi$ and $\mathsf{cr}(\pi^{\mathcal{T}}_{\xi,\xi'}) \geq \xi$.

Since the the sequence $\langle \zeta^i_\xi \mid i < \gamma \rangle$ is cofinal in τ_ξ, the computation in the previous paragraph yields $\pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi}[W \,||\, \tau_\xi] \subseteq \pi^{\mathcal{T}}_{0,\xi'}[W \,||\, \tau_{\xi'}]$. As the assignment $\zeta^i_\xi \mapsto \zeta^i_{\xi'}$ is cofinal in $\tau_{\xi'}$ (as we saw above), it follows that the set on the left is a cofinal subset of the set on the right. This completes the proof of Lemma 20. □

Without loss of generality we will assume that the conclusions of Lemma 20 hold for any pair of ordinals $\xi < \xi'$ in C^*. For any such pair of ordinals we let

$$\pi_{\xi,\xi'} = (\pi^{\mathcal{T}}_{0,\xi'})^{-1} \circ \pi^{\mathcal{U}}_{\xi,\xi'} \circ \pi^{\mathcal{T}}_{0,\xi} \restriction (W \,||\, \tau_\xi).$$

Then $\pi_{\xi,\xi'} : W \,||\, \tau_\xi \to W \,||\, \tau_{\xi'}$ is a fully elementary map such that

- $\mathsf{cr}(\pi_{\xi,\xi'}) = \xi$ and $\pi_{\xi,\xi'}(\xi) = \xi'$, and
- $\pi_{\xi,\xi'}$ maps $W \,||\, \tau_\xi$ cofinally into $W \,||\, \tau_{\xi'}$.

- The system $\langle W \,||\, \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ is commutative and continuous, so if ξ is a limit point of C^*, then $W \,||\, \tau_\xi$ is the direct limit of the system $\langle W \,||\, \tau_{\bar\xi}, \pi_{\bar\xi,\xi'} \mid \bar\xi < \xi' \text{ in } C^* \cap \xi \rangle$.

The above conclusions follow easily from the definition of $\pi_{\xi,\xi'}$ and the previous calculations. We give some details on the continuity: Given a limit point ξ of C^*, pick some $\bar\xi \in C^* \cap \xi$. If $x \in W \,||\, \tau_\xi$, pick some $\zeta < \tau_{\bar\xi}$ such that $x \in W \,||\, \pi_{\bar\xi,\xi}(\zeta)$; this is possible by the cofinality of the map $\pi_{\bar\xi,\xi}$. If $\bar f \in W \,||\, \tau_{\bar\xi}$ is a surjection of $\bar\xi$ onto $W \,||\, \zeta$, then $f = \pi_{\bar\xi,\xi}(\bar f)$ is a surjection of ξ onto $W \,||\, \pi_{\bar\xi,\xi}(\zeta)$, so $x = f(\alpha)$ for some $\alpha < \xi$. Now pick $\xi' \in C^* \cap [\bar\xi, \xi)$ such that $\alpha < \xi'$. Since f and α belong to $\mathrm{rng}(\pi_{\xi',\xi})$, it follows that $x \in \mathrm{rng}(\pi_{\xi',\xi})$ as well.

Lemma 22. *Let $\xi < \xi'$ be in C^*. Let $F_{\xi,\xi'}$ be the (ξ, ξ')-extender derived from $\pi_{\xi,\xi'}$. Then $F_{\xi,\xi'}$ is a W-extender that coheres to W.*

Proof. We verify that $F_{\xi,\xi'}$ coheres to W; the rest of the lemma is clear. As $W \,||\, \tau_\xi \models \mathsf{ZFC}^-$, the ultrapower $\mathsf{Ult}(W \,||\, \tau_\xi, F_{\xi,\xi'})$ can be elementarily embedded into $W \,||\, \tau_{\xi'}$ in the usual way by assigning $\pi_{\xi,\xi'}(f)(a)$ to the object in the ultrapower represented by $[a, f]$. This shows that the ultrapower is well-founded. Let Q be its transitive collapse, and let $k : Q \to W \,||\, \tau_{\xi'}$ be the embedding defined as above. We then have $\sigma \restriction (\xi' + 1) = \mathsf{id}$. It follows that $k = \mathsf{id}$, as otherwise $\mathsf{cr}(k)$ would be a cardinal in Q larger than ξ'. Hence Q is an initial segment of $W \,||\, \tau_{\xi'}$. Since $\pi_{\xi,\xi'}$ maps τ_ξ cofinally into $\tau_{\xi'}$ and $\pi_{\xi,\xi'} = k \circ i$, where $i : Q \to W \,||\, \tau_{\xi'}$ is the ultrapower embedding, then k maps $\mathbf{On} \cap Q$ cofinally into $\tau_{\xi'}$. It follows that $Q = W \,||\, \tau_{\xi'}$. □

Let

(3) $\quad \tilde W$ be the direct limit of the diagram $\langle W \,||\, \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \rangle$,

and

$$\pi_\xi : W \,||\, \tau_\xi \to \tilde W \text{ be the direct limit maps.} \tag{4}$$

Lemma 23. *Let θ be a regular cardinal larger than δ^+, and let X be an elementary substructure of H_θ such that $\kappa = X \cap \delta$ is an ordinal and*

$$\langle W \,||\, \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \textit{ in } C^* \rangle \in X.$$

Let H be the transitive collapse of X, and let $\sigma : H \to H_\theta$ be the inverse of the Mostowski collapsing isomorphism. Then $\kappa \in C^*$, $W \,||\, \tau_\kappa \in H$, $\sigma(W \,||\, \tau_\kappa) = \tilde W$, and $\sigma \restriction (W \,||\, \tau_\kappa) = \pi_\kappa$.

Proof. Since $C^* \in X$ and C^* is club in δ, the critical point κ of σ is a limit point of C^*, hence an element of C^*. Also $\sigma(C^* \cap \kappa) = C^*$. The inverse image of the diagram $\langle W \,||\, \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ under σ is thus of the form $\langle \bar W_\xi, \bar\pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \cap \kappa \rangle$ where $\sigma(\bar W_\xi) = W \,||\, \tau_\xi$ for all $\xi \in C^* \cap \kappa$. Since the height of each $\bar W_\xi$ is strictly smaller than κ, the map σ is the identity on $\bar W_\xi$, so $\bar W_\xi = W \,||\, \tau_\xi$ and $\bar\pi_{\xi,\xi'} = \pi_{\xi,\xi'}$ for all $\xi < \xi'$ in $C^* \cap \kappa$. Because the diagram $\langle W \,||\, \tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \rangle$ is continuous, the

structure $W\,||\,\tau_\kappa = \lim\langle W\,||\,\tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^* \cap \kappa\rangle$, being a direct limit of a diagram in H is itself an element of H, and the same is also true of the direct limit maps $\pi_{\xi,\kappa}$ where $\xi \in C^* \cap \kappa$. It follows that

$$\begin{aligned}\sigma(W\,||\,\tau_\kappa) &= \sigma(\lim\langle W\,||\,\tau_\xi, \pi_{\xi,\xi'}) \mid \xi < \xi' \text{ in } C^* \cap \kappa\rangle) \\ &= \lim\langle W\,||\,\tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^*\rangle = \tilde{W}.\end{aligned}$$

Furthermore, from the elementarity of σ, we obtain that $\sigma(\pi_{\xi,\kappa}) = \pi_\xi$ for all ordinals $\xi \in C^* \cap \kappa$. Thus if $x \in W\,||\,\tau_\kappa$, then $x = \pi_{\xi,\kappa}(\bar{x})$ for some $\xi \in C^* \cap \kappa$ and $\bar{x} \in W\,||\,\tau_\xi$, hence

$$\sigma(x) = \sigma(\pi_{\xi,\kappa}(\bar{x})) = \sigma(\pi_{\xi,\kappa})(\sigma(\bar{x})) = \pi_{\xi,\delta}(\bar{x}) = \pi_{\kappa,\delta}(\pi_{\xi,\kappa}(\bar{x})) = \pi_{\kappa,\delta}(x),$$

which shows that $\sigma \restriction W\,||(\tau_\kappa) = \pi_\kappa$. □

It follows from the above lemma that the collection of all elementary substructures X of H_θ such that $\kappa = X \cap \delta$ and $\tilde{W}$ (from (3)) collapses to $W\,||\,\kappa^{+W}$ under the Mostowski collapsing isomorphism coming from X, is a club subset of $\mathcal{P}_\delta(H_\theta)$. This is a strengthening of Mitchell-Schindler [MS04, Lemma 3.5].

We do not know if there is a significanly simpler proof of Lemma 23 based on the idea used in Mitchell-Schindler [MS04, Lemma 3.5], as it seems that one needs to require $\tilde{W} \in X$ in order to obtain the conclusions of Lemma 23.

Given a regular cardinal $\theta > \delta^+$ and an elementary substructure X of H_θ such that $X \cap \delta$ is an ordinal and the diagram $\langle W\,||\,\tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{ in } C^*\rangle$ is an element of X, define the following objects:

- H_X is the transitive collapse of X and $\sigma_X : H_X \to X$ is the inverse of the Mostowski collapsing isomorphism.
- $\kappa_X = \mathsf{cr}(\sigma_X)$ and $\tau_X = \tau_{\kappa_X} = \kappa_X^{+W}$.
- If $Y \supseteq X$ is an elementary substructure of H_θ such that $\kappa_Y > \kappa_X$, then $\sigma_{X,Y} : H_X \to H_Y$ is defined by $\sigma_{X,Y} = \sigma_Y^{-1} \circ \sigma_X$.
- If X, Y are as above, we write π_X for π_{κ_X} and $\pi_{X,Y}$ for π_{κ_X,κ_Y}.

Here of course all objects introduced above depend on θ, but in our applications we will keep θ fixed, so we treat it as a suppressed parameter. It follows from the above lemma that $\sigma_X \restriction (W\,||\,\tau_X) = \pi_X$ and $\sigma_{X,Y} \restriction (W\,||\,\tau_X) = \pi_{X,Y}$, so these restrictions depend only on the ordinals κ_X and κ_Y and not on the structures X and Y themselves. In particular we get

$$F_{X,Y} \stackrel{\text{def}}{=} F_{\kappa_X,\kappa_Y} = \text{the } W\text{-extender at } (\kappa_X, \kappa_Y) \text{ derived from } \sigma_{X,Y}, \tag{5}$$

and

$$F_X \stackrel{\text{def}}{=} \text{the } W\text{-extender at } (\kappa, \delta) \text{ derived from } \pi_X = \sigma_X \restriction (W\,||\,\tau_X). \tag{6}$$

Lemma 24. *Given a stationary set S as in* (2)*, let S^θ be the collection of all $X \subseteq H_\theta$ satisfying the following requirements:*

(a) *X is an elementary substructure of H_θ.*
(b) *The diagram $\langle W\,||\,\tau_\xi, \pi_{\xi,\xi'} \mid \xi < \xi' \text{in } C^*\rangle$ is an element of X.*

(c) $X \cap \delta^+ \in S$.

Then S^θ is stationary, and for any $X, Y \in S^\theta$ such that $X \subseteq Y$ and $\kappa_X < \kappa_Y$, the extender $F_{X,Y}$ is an element of $\mathbf{M}$.

Proof. The stationarity of S^θ follows from the stationarity of S and from the fact that the collection of all X satisfying clauses (a) and (b) is club in $\mathcal{P}(H_\theta)$. We now verify that $F_{X,Y} \in \mathbf{M}$. For this, it will suffice to see that $\pi_{X,Y} \in \mathbf{M}$. We first observe that $\pi_{X,Y} \restriction \tau_X \in \mathbf{M}$; for this we use the fact that $\pi_{X,Y} \restriction \tau_X = (\pi_Y^{-1} \restriction \tau_Y) \circ (\pi_X \restriction \tau_X)$. To see that both restriction maps on the right of this equality are in $\mathbf{M}$, let us argue for instance for X. Since $X \in S^\theta$ and $S \subseteq \mathbf{M}$, the restriction $\pi_X \restriction \mathsf{otp}(X \cap \delta^+) = \sigma_X \restriction \mathsf{otp}(X \cap \delta^+)$, being the (unique) isomorphism between $\mathsf{otp}(X \cap \delta^+)$ and $X \cap \delta^+$, is an element of $\mathbf{M}$. The equality here comes from Lemma 23. As $\tau_X \leq \mathsf{otp}(X \cap \delta^+)$, the restriction $\pi_X \restriction \tau_X$ is an element of $\mathbf{M}$ as well.

Now, since $W \,||\, \tau_X$ and $W \,||\, \tau_Y$ are both in $\mathbf{M}$, it suffices to argue that $\pi_{X,Y}$ is fully determined by $\pi_{X,Y} \restriction \tau_X$. Since $W \,||\, \tau_X \models \mathsf{ZFC}^-$, the canonical well-ordering $<_W$ orders $W \,||\, \tau_X$ in order-type τ_X. So if $a \in W \,||\, \tau_X$, then there is an ordinal $\alpha < \tau_X$ such that a is the α-th element of $W \,||\, \tau_X$ under $<_W$. As $\pi_{X,Y}$ is elementary, $\pi_{X,Y}(a)$ is the $\pi_{X,Y}(\alpha)$-th element of $W \,||\, \tau_Y$ under $<_W$. So the value of $\pi_{X,Y}(a)$ is fully determined by the action of $\pi_{X,Y}$ on the elements of τ_X. $\square$

3.3. **Proof of Theorem 5.(a).** So far we completed task (A) in the strategy outlined at the beginning of this section. We now proceed toward completing task (B). Our argument here depends on the anti-large cardinal hypothesis we make in $\mathbf{V}$. We start with the case where no proper class inner model with a Woodin cardinal in the sense of $\mathbf{V}$ is allowed, that is, we are about to complete the proof of Theorem 5.(a).

Here we apply a frequent extension argument, similar to that in Schimmerling-Steel [SS99]; see also Mitchell-Schimmerling [MS95]. A general version of the frequent extension argument in the simplified context of extender models below $0^\P$ is described in Zeman [Zem02, §§7.5, 8.3], and also Cox [Cox09]. The layout of the argument in Cox [Cox09] uses a system of extender models indexed by structures from a suitable stationary set, rather then a linear chain of internally aproachable structures. In our situation, it will be convenient to use this layout. The initial step of the frequent extension argument thus produces a system of structures and embeddings indexed with elements X of the collection S^θ_δ for a suitable θ (see below for the definition of S^θ_δ); we denote these objects by $\bar{N}_X, N^*_X, \rho_X, \tilde{\sigma}_X, \sigma'_X, \ldots$ We construct these objects in such a way that for each index X, each individual $\bar{N}_X, N^*_X, \ldots$ is an element of $\mathbf{M}$. The entire indexed system, of course, need not be an element of $\mathbf{M}$. It may be the case (we do not know) that one can proceed more liberally and run some constructions in $\mathbf{V}$ instead of $\mathbf{M}$, but in the end we need to guarantee that certain premice and phalanxes we construct are in $\mathbf{M}$, and are iterable in the sense of $\mathbf{M}$. Our approach

guarantees this automatically. Also, it is worth noting that in order to run the frequent extension argument, we merely need the anti-large cardinal hypothesis that no proper class inner model with a Woodin cardinal exists in the sense of $\mathbf{M}$. (So at this point in the proof of Theorem 5.(a) we do not need the full anti-large assumption of the theorem.)

Recall from (1) that S_δ is the collection of all sets $x \in \mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ such that $\mathsf{cf}^{\mathbf{M}}(x \cap \delta) > \omega$. Working in $\mathbf{V}$, pick a regular $\theta > \delta^+$. Following the notation introduced in Lemma 24, the set S^θ_δ is the collection of all sets $X \subseteq H_\theta$ satisfying (a)–(c) in Lemma 24 with S_δ in place of S.

We show that for all but nonstationarily many $X \in S^\theta_\delta$ and all $Y \in S^\theta_\delta$ such that $X \in Y$, the ultrapower $\mathsf{Ult}(W, F_{X,Y})$ is well-founded and the phalanx $(W, \mathsf{Ult}(W, F_{X,Y}), \kappa_Y)$ is normally iterable in the sense of $\mathbf{M}$. (Under our anti-large cardinal hypothesis, this is actually equivalent to its iterability in the sense of $\mathbf{V}$). By Lemma 16 applied inside $\mathbf{M}$, the extender $F_{X,Y}$ is an element of W. However, by Lemma 22, the extender induces the ultrapower map $\pi_{X,Y}$ that maps τ_X cofinally into τ_Y, which is false in W. This is a contradiction, and completes the proof of Theorem 5.(a).

Remark 25. *It is the use of the frequent extension argument that necessitates the assumption of stationarity of S_δ in place of the weaker assumption that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. In general, the assumption that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M} - S_\delta$ is stationary is not sufficient to run the frequent extension argument; see for instance Räsch-Schindler* [RS05].

Before we proceed, recall that given phalanxes (P, P', α) and (Q, Q', β), a pair of maps (σ, σ') is an embedding of (P, P', α) into (Q, Q', β) if and only if $\sigma : P \to P'$ and $\sigma' : Q \to Q'$ are Σ_0-elementary and cardinal preserving, $\sigma \restriction \alpha = \sigma' \restriction \alpha$, $\sigma[\alpha] \subseteq \beta$, and $\sigma'(\alpha) \geq \beta$. We will only deal with embeddings that are fully elementary, but of course one may consider embeddings of various degrees of preservation.

If a normal iteration tree $\mathcal{T}$ on (P, P', α) picks unique branches, we can copy $\mathcal{T}$ onto a normal iteration tree on (Q, Q', β) via the embedding (σ, σ'). Note also that we will use this terminology also in the case where $Q = Q'$, that is, when we talk about an embedding of a phalanx into a premouse.

For each $X \in S^\theta_\delta$, pick some $Y = Y(X)$ such that $X \in Y$. Then let $G_X = F_{X,Y(X)}$ and $\lambda_X = \kappa_{Y(X)}$. We will prove that for all but nonstationarily many $X \in S^\theta_\delta$,

(7)

$\mathsf{Ult}(W, G_X)$ is well-founded and $(W, \mathsf{Ult}(W, G_X), \lambda_X)$ is normally iterable.

Here we talk about iterability in the sense of $\mathbf{M}$.

Heading for a contradiction, assume there is a stationary

$$S \subseteq S^\theta_\delta \tag{8}$$

such that either $\mathsf{Ult}(W, G_X)$ is ill-founded or else this ultrapower is well-founded but the phalanx $(W, \mathsf{Ult}(W, G_X), \lambda_X)$ is not normally iterable in the sense of $\mathbf{M}$. So there is some $\theta_0 > \delta^+$ in $\mathbf{V}$ such that θ_0 is a successor

cardinal in W and the same is true with $W \,||\, \theta_0$ in place of W for all $X \in S$. Write N for $W \,||\, \theta_0$ and N'_X for $\mathsf{Ult}(N, G_X)$, where we have identified well-founded parts with their transitive collapses. Thus even if N'_X is ill-founded, it is well-founded past λ_X, and if N'_X is well-founded, then it is actually transitive. For each $X \in S$ let $\mathcal{T}_X \in \mathbf{M}$ be a putative normal iteration tree on the phalanx (N, N'_X, λ_X) witnessing that the phalanx is not normally iterable in $\mathbf{M}$. Here we mean that either $\mathcal{T}_X$ has a last ill-founded model, or $\mathcal{T}_X$ is of limit length and has no cofinal well-founded branch. If N'_X is ill-founded, we let $\mathcal{T}_X$ be the phalanx (N, N'_X, λ_X) whose last model is N'_X, which offers uniform treatment of the situations listed above.

Pick a $\mathbf{V}$-regular $\theta^* > \theta$ such that $\theta, S, \mathcal{T}^X \in H_{\theta^*}$ whenever $X \in S$. For each X, construct an elementary substructure $Z_X \prec H^{\mathbf{M}}_{\theta^*}$ such that

- $Z_X \in \mathbf{M}$ and is countable in $\mathbf{M}$, and
- $G_X, \mathcal{T}_X, \tilde{\tau} \in Z_X$, where $\tilde{\tau} = \delta^{+\tilde{W}}$.

Let H^Z_X be the transitive collapse of Z_X, let $\rho_X : H^Z_X \to H^{\mathbf{M}}_{\theta^*}$ be the inverse to the associated Mostowski collapsing isomorphism, and let

$$(\bar{\mathcal{T}}_X, \bar{N}_X, \bar{N}'_X, \bar{\lambda}_X) = \rho_X^{-1}(\mathcal{T}_X, N, N'_X, \lambda_X).$$

So each of these objects individually is in $\mathbf{M}$, although the collection consisting of all of them may not be an element of $\mathbf{M}$.

Next we do a pressing down argument which is a typical part of any frequent extension argument. Consider $X \in S$ such that $H^{\mathbf{M}}_{\delta^+} \in X$. Recall that since $X \cap \delta^+ \in \mathbf{M}$, the restriction $\sigma_X \restriction \tau_X$ is an element of $\mathbf{M}$ as well. Since δ is a cardinal and $\sigma_X \restriction \tau_X$ maps $\tau_X < \delta$ cofinally into $\tilde{\tau}$, necessarily $\tilde{\tau} < \delta^{+\mathbf{M}}$, so there is a surjection $f : \delta \to \tilde{\tau}$ such that $f \in \mathbf{M}$. If X' is an elementary substructure of H_{θ^*} such that $X = X' \cap H_\theta$ then, since $\tilde{\tau}, H^{\mathbf{M}}_{\delta^+} \in X \subseteq X'$, there is a surjection $f : \delta \to \tilde{\tau}$ such that $f \in \mathbf{M} \cap X'$. It follows that $\sigma_X[Z_X \cap \tau_X] \subseteq \mathbf{M} \cap X \subseteq \mathbf{M} \cap X'$, hence $\sigma_X[Z_X \cap \tau_X] \subseteq f[\kappa_X]$.

Since $\sigma_X \restriction \tau_X$ is an element of $\mathbf{M}$, so is $\sigma_X[Z_X \cap \tau_X]$. In $\mathbf{M}$, the set $\sigma_X[Z_X \cap \tau_X]$ is a countable subset of $\tilde{\tau}$. Since $\mathsf{cf}^{\mathbf{M}}(\kappa_X)$ is uncountable in $\mathbf{M}$, we can find an ordinal $\alpha < \kappa_X$ such that $\sigma_X[Z_X \cap \tau_X] \subseteq f[\alpha]$. But since $f \in X' \cap H^{\mathbf{M}}_{\delta^+} \subseteq X$, also $f[\alpha]$ is an element of X. Thus, letting $a = f[\alpha]$, the set X witnesses the existential quantifier in the statement

$$H_{\theta^*} \models (\exists v \in S)(a \in v). \tag{9}$$

Obviously the set

$$S' = \{X \in S \mid H^{\mathbf{M}}_{\delta^+} \in X \text{ and } (\exists X' \prec H_{\theta^*})(S \in X' \;\&\; X' \cap H_\theta = X)\}$$

is stationary. Given $X \in S'$, let $X' \prec H_{\theta^*}$ witness this. Since $a, S \in X'$, applying the elementarity of X' to (9) yields

$$X' \models (\exists v \in S)(a \in v).$$

So there is some $\bar{X} \in S \cap X'$ with $a \in \bar{X}$, and such $\bar{X}$ is obviously an element of X. Since $\sigma_X[Z_X \cap \tau_X] \subseteq a \subseteq \bar{X}$ (the latter inclusion follows from the facts that $a \in \bar{X}$, the cardinality of a is less than δ, and $\bar{X} \cap \delta$ is transitive), we

can define a regressive function $g : S' \to S'$ such that $\sigma_X[Z_X \cap \tau_X] \subseteq g(X)$ for all $X \in S'$. By pressing down, there are a stationary $S^* \subseteq S'$ and an $X^* \in S$ such that

$$\sigma_X[Z_X \cap \tau_X] \subseteq X^* \text{ for all } X \in S^*. \tag{10}$$

Fix the following notation:

- H^* is the transitive collapse of X^*.
- $\sigma^*_X : H^Z_X \to H^*$ is a partial map defined by $\sigma^*_X = \sigma^{-1}_{X^*,X} \circ \rho_X$.
- $\tau^*_X = \sup(\sigma^*_X[\bar{\tau}_X])$, where $\bar{\tau}_X = \rho^{-1}_X(\tau_X)$.

It follows from (10) that $\rho_X[\bar{\tau}_X] = Z_X \cap \tau_X \subseteq \sigma^*_X[\tau_{X^*}]$, so $\sigma^*_X(\xi)$ is defined for all $\xi < \bar{\tau}_X$. Furthermore, since $\rho_X, \sigma_{X^*,X} \restriction \tau_{X^*}$ are both elements of $\mathbf{M}$, so is $\sigma^*_X \restriction \bar{\tau}_X$. An argument similar to that in the proof of Lemma 24 shows that $\sigma^*_X \restriction (\bar{N}_X \,||\, \bar{\tau}_X)$ is fully determined by $\sigma^*_X \restriction \bar{\tau}_X$, and is defined on the entire $\bar{N} \,||\, \bar{\tau}_X$. Hence $\sigma^*_X \restriction (\bar{N}_X \,||\, \bar{\tau}_X) : \bar{N}_X \,||\, \bar{\tau}_X \to W \,|\, \tau^*_X = N \,|\, \tau^*_X$ is a Σ_0-preserving embedding mapping $\bar{N}_X \,||\, \bar{\tau}_X$ cofinally into $N \,|\, \tau^*_X$. (Recall that $N \,|\, \tau^*_X$ is the same as $N \,||\, \tau^*_X$ without the top extender; and since $\bar{\tau}_X$, being a cardinal in $\bar{N}_X$, does not index an extender in $\bar{N}_X$, under a slight abuse of notation we have $\bar{N}_X \,||\, \bar{\tau}_X = \bar{N}_X \,|\, \bar{N}_X$.) With a tiny bit of effort one can see that $\sigma^*_X \restriction (\bar{N}_X \,||\, \bar{\tau}_X)$ is fully elementary. We thus have the following conclusion:

$$\sigma^*_X \in \mathbf{M} \text{ and } \sigma_{X^*,X} \circ \sigma^*_X \restriction (\bar{N}_X \,||\, \bar{\tau}_X) = \rho_X \restriction (\bar{N}_X \,||\, \bar{\tau}_X). \tag{11}$$

This agreement of $\sigma_{X^*,X} \circ \sigma^*_X$ with ρ_X on $\bar{N}_X \,||\, \bar{\tau}_X$ makes it possible to run the argument in the proof of the Interpolation Lemma (Zeman [Zem02, Lemma 3.6.10]), and obtain an acceptable structure N^*_X extending $N \,|\, \tau^*_X$ and embeddings $\tilde{\sigma}_X : \bar{N}_X \to N^*_X$ and $\sigma'_X : N^*_X \to N$ with the following properties:

- N^*_X is an end-extension of $N \,|\, \tau^*_X$.
- $\tilde{\sigma}_X$ is an extension of $\sigma^*_X \restriction \bar{N}_X$, and σ'_X is an extension of $\sigma_{X^*,X} \restriction (N \,|\, \tau^*_X)$.
- Both maps $\tilde{\sigma}_X$ and σ'_X are Σ_0-preserving, and cardinal preserving.
- $\sigma'_X \circ \tilde{\sigma}_X = \rho_X \restriction \bar{N}_X$.
- N^*_X, σ'_X and $\tilde{\sigma}_X$ are elements of $\mathbf{M}$.

Roughly speaking, N^*_X is constructed as the ultrapower (Zeman [Zem02] uses the term "pseudoultrapower") of $\bar{N}_X$ by σ^*_X, the map $\tilde{\sigma}_X$ is the ultrapower embedding, and σ'_X is the factor map between $\tilde{\sigma}_X$ and $\rho_X \restriction \bar{N}_X$. The last item follows from the fact that the entire ultrapower construction takes place in $\mathbf{M}$. Since N is an initial segment of an extender model below its successor cardinal, N is a passive premouse and $N \models \mathsf{ZFC}^-$. It follows that both maps are in fact fully elementary, and N^*_X is a passive premouse. Also, we can view σ'_X as a map from N^*_X into W, and in this case σ'_X is Σ_0-preserving and cardinal preserving.

The above construction gives us the following for all $X, Y \in S^*$ such that $Y = Y(X)$ or $Y(X) \in Y$:

$$\text{Either } \mathsf{Ult}(N^*_X, F_{X^*,Y}) \text{ is ill-founded,} \quad \text{or else } (W, \mathsf{Ult}(N^*_X, F_{X^*,Y}), \kappa_Y) \text{ is not iterable.} \tag{12}$$

The conclusion for the case where $Y(X) \in Y$ follows immediately from the conclusion for $Y = Y(X)$ since the pair (id, k) is an embedding of the phalanx $(W, \mathsf{Ult}(W, F_{X^*,Y(X)}), \lambda_X)$ into $(W, \mathsf{Ult}(W, F_{X^*,Y}), \kappa_Y)$ where k is the factor map between the two ultrapower embeddings. To see the conclusion for $Y = Y(X)$, recall that we put $\mathcal{T}_X$, an iteration tree witnessing the non-iterability of the phalanx $(N, \mathsf{Ult}(N, F_{X^*,Y}), \lambda_X)$, into Z_X. Then an argument similar to the proof of Steel [Ste96, Lemma 2.4(b)] yields that $\bar{\mathcal{T}}_X$ witnesses that $(\bar{N}_X, \bar{N}'_X, \bar{\lambda}_X)$ is not iterable in the sense of $\mathbf{M}$; here recall that $\bar{N}'_X = \mathsf{Ult}(\bar{N}_X, \bar{G}_X)$ where $\bar{G}_X = \rho_X^{-1}(G_X)$. (We stress that we are assuming $Y = Y(X)$.) Here is the place where we use the fact that θ_0 is a successor cardinal in W; see Steel [Ste96, Lemma 6.13] for details concerning why this assumption is useful.

Let $k' : \bar{N}'_X \to \mathsf{Ult}(N^*_X, F_{X^*,Y})$ be the map defined by

$$k' : [a, f]_{\bar{G}_X} \mapsto [\rho_X(a), \tilde{\sigma}_X(f)]_{F_{X^*,Y}}$$

whenever $a \in [\bar{\lambda}_X]^{<\omega}$ and $f \in \bar{N}_X$ is such that $\mathrm{dom}(f) = [\bar{\kappa}_X]^{|a|}$, where $\bar{\kappa}_X = \mathsf{cr}(\bar{G}_X)$. Then the pair $(\rho_X \restriction \bar{N}_X, k')$ is an embedding of the phalanx $(\bar{N}_X, \bar{N}'_X, \bar{\lambda}_X)$ into $(W, \mathsf{Ult}(N^*_X, F_{X^*,Y}), \lambda_X)$ witnessing (12) in the case where $Y = Y(X)$. This follows by the standard computation: First notice that it follows from the definition of k', for any $a \in [\bar{\lambda}_X]^{<\omega}$ we have $k'(a) = k'([a, \mathsf{id}]_{\bar{G}_X}) = [\rho_X(a), \mathsf{id}]_{F_{X^*,Y}} = \rho_X(a)$ and $k'(\bar{\lambda}_X) = \lambda_X$, so we have the necessary agreement between the two maps. To see that k' is sufficiently elementary, given a formula $\varphi(v)$, Łoś theorem yields $\bar{N}'_X \models \varphi([a, f])$ iff

$$x = \{u \in [\bar{\kappa}_X]^{|a|} \mid \bar{N}_X \models \varphi(f(u))\} \in (\bar{G}_X)_a.$$

By applying ρ_X, we have $\rho_X(x) \in (G_X)_{\rho_X(a)}$ or, equivalently, $\rho_X(a) \in \sigma_{X,Y}(\rho_X(x))$. Since $\sigma_{X,Y}(\rho_X(x)) = \sigma_{X,Y} \circ \sigma'_X \circ \sigma^*_X(x) = \sigma_{X^*,Y}(\sigma^*_X(x))$, we conclude that $\sigma^*_X(x) \in (F_{X^*,Y})_{\rho(a)}$. Now

$$\sigma^*_X(x) = \{u \in [\kappa_{X^*}]^{|a|} \mid N^*_X \models \varphi(\tilde{\sigma}_X(f)(u))\}$$

and, by another application of Łoś theorem, we have

$$\mathsf{Ult}(N^*_X, F_{X^*,Y}) \models \varphi([\rho_X(a), \tilde{\sigma}_X(f)].$$

This completes the proof of (12).

From now on, the argument closely follows Mitchell-Schimmerling [MS95, §3]. Write κ^* for κ_{X^*}. We introduce a relation $<_{\mathsf{S}}$ on premice Q such that

(W, Q, κ^*) is an iterable phalanx:

(13) $Q <_{\mathsf{S}} Q'$ iff there is a normal iteration tree $\mathcal{T}$ on (W, Q, κ^*) such that Q' is an initial segment of the last model $M^{\mathcal{T}}_\infty$ of $\mathcal{T}$, and, letting b be the main branch of $\mathcal{T}$, one of the following holds:

- W is on b.
- Q is on b and there is a truncation point on b.
- Q is on b, there is no truncation point on b, and Q' is a proper initial segment of $M^{\mathcal{T}}_\infty$.

The following lemma is a simple instance of Mitchell-Schimmerling [MS95, Lemma 3.2], rephrased for our purposes.

Lemma 26. *Let $\mathfrak{Q}$ be the transitive closure of the singleton $\{W\}$ under the relation $<_{\mathsf{S}}$. Then $<_{\mathsf{S}}$ restricted to $\mathfrak{Q}$ is well-founded.*

Proof. See the proof of Mitchell-Schimmerling [MS95, Lemma 3.2]. □

The above lemma says that if we start with the phalanx (W, W, κ^*) and build a linear chain of iteration trees $\mathcal{T}_i$ such that $\mathcal{T}_0$ is on (W, W, κ^*) and $\mathcal{T}_{i+1}$ is on (W, Q_{i+1}, κ^*), where Q_{i+1} is the last model of $\mathcal{T}_i$, then for some finite n the model Q_{n+1} is on the main branch of $\mathcal{T}_n$ and there is no truncation on the branch.

The lemma is formulated for $\mathbf{V}$, but we apply it in $\mathbf{M}$ for each individual $X \in S^*$. Given $X \in S^*$, either $\mathsf{Ult}(W, F_{X^*,X})$ is ill-founded or else $(W, \mathsf{Ult}(W, F_{X^*,X}), \kappa_X)$ is not iterable; this follows from our initial assumption on S. By Lemma 26 applied in $\mathbf{M}$, for every $X \in S^*$ there is a $<_{\mathsf{S}}$-minimal Q_X such that the phalanx (W, Q_X, κ^*) is iterable, and either $\mathsf{Ult}(Q_X, F_{X^*,X})$ is ill-founded or else $(W, \mathsf{Ult}(Q_X, F_{X^*,X}), \kappa_X)$ is not iterable. So $Q_X \in \mathbf{M}$, and the notion of iterability is also in the sense of $\mathbf{M}$.

The next lemma says that it is possible to find a structure $X \in S^*$ such that Q_X can replace Q_Y for club many structures Y.

Lemma 27. *There is an $X \in S^*$ such that for all $Y \in S^*$ with $X \in Y$, the following hold in $\mathbf{M}$:*

(a) *Either $\mathsf{Ult}(Q_X, F_{X^*,Y})$ is ill-founded or else $(W, \mathsf{Ult}(Q_X, F_{X^*,Y}), \kappa_Y)$ is not iterable.*

(b) *The premouse Q_X is $<_{\mathsf{S}}$-minimal, that is, if $Q' <_{\mathsf{S}} Q_X$ then* (a) *fails with Q' in place of Q_X.*

Proof. Notice that if $X \in Y$, and either $\mathsf{Ult}(Q_X, F_{X^*,X})$ is ill-founded, or

$$(W, \mathsf{Ult}(Q_X, F_{X^*,X}, \kappa_X)$$

is not iterable, then (a) holds, that is, either $\mathsf{Ult}(Q_X, F_{X^*,Y})$ is ill-founded, or else the phalanx

$$(W, \mathsf{Ult}(Q_X, F_{X^*,Y}), \kappa_Y)$$

is not iterable.

This follows from the fact that the extender $F_{X^*,Y}$ is "larger" than $F_{X^*,X}$. More precisely, let

$$k : \mathsf{Ult}(Q_X, F_{X^*,X}) \to \mathsf{Ult}(Q_X, F_{X^*,Y})$$

be the factor map. Then the pair (id, k) is an embedding of the phalanx $(W, \mathsf{Ult}(Q_X, F_{X^*,X}), \kappa_X)$ into the phalanx $(W, \mathsf{Ult}(Q_X, F_{X^*,Y}), \kappa_Y)$. Informally, since Y is "larger" than X, the structure Q_X is still "bad" in the sense that it satisfies (a), but need not be $<_{\mathsf{S}}$-minimal.

Assuming the lemma is false, for every $X \in S^*$ there is some $Y \in S^*$ with $X \in Y$, such that Q_X is not minimal such that (a) holds. Letting $X_0 = X$, construct inductively a sequence $\langle X_i \mid i \in \omega \rangle$ such that $X_i \in X_{i+1} \in S^*$ and $Q_{X_{i+1}} <_{\mathsf{S}} Q_{X_i}$. That is, there is an iteration tree $\mathcal{T}_i \in \mathbf{M}$ on (W, Q_{X_i}, κ) such that $Q_{X_{i+1}}$ is the last model of $\mathcal{T}_i$, and either W is on the main branch of $\mathcal{T}_i$, or else Q_{X_i} is on the main branch of $\mathcal{T}_i$, in which case either there is a truncation on the main branch of $\mathcal{T}_i$, or else $Q_{X_{i+1}}$ is a proper initial segment of the last model of $\mathcal{T}_i$. The sequence $\langle Q_{X_i} \mid i \in \omega \rangle$ witnesses that the relation $(<_{\mathsf{S}})^{\mathbf{M}}$ is ill-founded in the sense of $\mathbf{V}$. By absoluteness of well-foundedness, $(<_{\mathsf{S}})^{\mathbf{M}}$ is ill-founded in the sense of $\mathbf{M}$, a contradiction. $\square$

Now pick X as in Lemma 27, and $Y \in S^*$ such that $X^*, X, Y(X) \in Y$. Working in $\mathbf{M}$, compare the phalanxes (W, Q_X, κ^*) and (W, N^*_X, κ^*). Let $\mathcal{U}$ and $\mathcal{V}$ be the iteration trees arising in the comparison. That is, $\mathcal{U}$ is on (W, Q_X, κ^*).

Because W witnesses the soundness of $\mathbf{K}^{\mathbf{M}} \,||\, \delta^+$, the standard argument shows that W cannot be on the main branches of both trees. We now get a final contradiction by ruling out all other possibilities.

We begin with the observation that Q_X must be on the main branch of $\mathcal{U}$. Otherwise, N^*_X is on the main branch of $\mathcal{V}$, as follows from the previous paragraph. Furthermore, since W is on the main branch of $\mathcal{U}$, and W computes successor cardinals correctly on a stationary class of cardinals, there is no truncation on the main branch of $\mathcal{V}$. Letting R be the last model on $\mathcal{V}$, we have $R <_{\mathsf{S}} Q_X$, which by Lemma 27 means that $\mathsf{Ult}(R, F_{X^*,Y})$ is well-founded and $(W, \mathsf{Ult}(R, F_{X^*,Y}), \kappa_Y)$ is iterable. On the other hand, by (12) we have that either $\mathsf{Ult}(N^*_X, F_{X^*,Y})$ is ill-founded or else $(W, \mathsf{Ult}(N^*_X, F_{X^*,Y}), \kappa_Y)$ is not iterable. Moreover, let

$$k : \mathsf{Ult}(N^*_X, F_{X^*,Y}) \to \mathsf{Ult}(R, F_{X^*,Y})$$

be the map defined by $k([a, f]) = [a, \pi^{\mathcal{V}}(f) \restriction [\kappa^*]^{|a|}]$, where $\pi^{\mathcal{V}} : N^*_X \to R$ is the map along the main branch of $\mathcal{V}$. Then the pair (id, k) is an embedding of the phalanx $(W, \mathsf{Ult}(N^*_X, F_{X^*,Y}), \kappa_Y)$ into $(W, \mathsf{Ult}(R, F_{X^*,Y}), \kappa_Y)$. This is a contradiction, which completes the proof that Q_X must be on the main branch of $\mathcal{U}$.

Let Q be the last model on $\mathcal{U}$, and let R, as above, be the last model on $\mathcal{V}$. We next argue that there is no truncation on the main branch of $\mathcal{U}$ and $Q \trianglelefteq R$. Otherwise, a simple discussion by cases yields that there is no truncation on the main branch of $\mathcal{V}$, and $R \trianglelefteq Q$. Moreover, $R <_{\mathsf{S}} Q_X$.

Notice also that Q_X is a set, because any counterexample to iterability is witnessed by a set-sized initial segment of the model. From this it follows that N^*_X is on the main branch of $\mathcal{V}$. Now it is easy to see that we can get a contradiction as in the previous paragraph.

To summarize, we are left with the following situation:

- Q_X is on the main branch of $\mathcal{U}$.
- There is no truncation on the main branch of $\mathcal{U}$, and $Q \trianglelefteq R$.

Recall that the pair (id, σ'_X) is an embedding of the phalanx (W, N^*_X, κ^*) into the model W. The map σ'_X is defined below (11), and when viewed as a map from N^*_X into W, it is Σ_0-preserving and cardinal-preserving. (Here we again use the fact that θ_0 was chosen to be a cardinal in W.) Let $\mathcal{V}'$ be the copy of $\mathcal{V}$ via the pair (id, σ'_X), let R' be the last model of $\mathcal{V}'$, and let $\sigma' : R \to R'$ be the copy map. Also, let $\pi^{\mathcal{U}} : Q_X \to Q$ be the iteration map along the main branch of $\mathcal{U}$. Now $R \,|\, \tau_R = N^*_X \,|\, \tau_R$, where $\tau_R = (\kappa^*)^{+R}$ and, by the copying construction, $\sigma' \restriction (R \,|\, \tau_R) = \sigma'_X \restriction (R \,|\, \tau_R)$. It follows that the extender on R derived from σ'_X is compatible with $F_{X^*,X}$ and measures all sets in $\mathcal{P}(\kappa^*) \cap Q$.

Let $\tilde{R} = \sigma'(Q)$; if $Q = R$, we of course let $\tilde{R} = R'$. Define a map $k : \mathsf{Ult}(Q_X, F_{X^*,X}) \to \tilde{R}$ by

$$k : [a, f] \mapsto [a, \sigma' \circ \pi^{\mathcal{U}}(f) \restriction [\kappa^*]^{|a|}]$$

for $a \in [\kappa_X]^{<\omega}$ and $f \in Q_X$ such that $\mathrm{dom}(f) = [\kappa^*]^{|a|}$.

Using Łoś theorem, a straightforward computation similar to that in the proof of (12) shows that k is Σ_0-preserving. It is also easy to see that k is cardinal preserving. Immediately from the definition of k we see that $k \restriction \kappa_X = \mathsf{id}$. It follows that the pair (id, k) is an embedding of the phalanx $(W, \mathsf{Ult}(Q_X, F_{X^*,X}), \kappa_X)$ into the phalanx $(W, \tilde{R}, \kappa_X)$. Since the iteration indices in $\mathcal{V}$ are above κ^*, the iteration indices in $\mathcal{V}'$ are above $\sigma'_X(\kappa^*) = \kappa_X$, so $(W, \tilde{R}, \kappa_X)$ is a W-generated phalanx in the sense of Steel [Ste96, Definition 6.7]. As W is embeddable into $\mathbf{K}^c$, the phalanx $(W, \tilde{R}, \kappa_X)$ is embeddable into a $\mathbf{K}^c$-generated phalanx, and is therefore iterable (see Steel [Ste96, Lemma 6.9]). It follows that also $(W, \mathsf{Ult}(Q_X, F_{X^*,X}), \kappa_X)$ is iterable, which contradicts the definition of Q_X.

Thus, assuming that $\mathsf{Ult}(W, G_X)$ is ill-founded or else $(W, \mathsf{Ult}(W, G_X), \lambda_X)$ fails to be iterable for all $X \in S$ where S comes from (8), we arrived at a final contradiction, thereby completing the proof of Theorem 5.(a).

3.4. **Proof of Theorem 5.(b).** Finally we turn to the proof of Theorem 5.(b). Here we work under the assumption that $0^{\P}$ does not exist in $\mathbf{M}$, which means that, if we use premice with Jensen's indexing of extenders, as described in Zeman [Zem02, Chapter 8], then the comparison process can be carried out using just linear iterations. In particular, $\mathbf{K}^{\mathbf{M}}$ is always linearly iterable in the sense of $\mathbf{V}$, no matter how large premice exist in $\mathbf{V}$. Linear iteration trees along with the condensation properties of such premice allow

us, for an extender with support below δ^+, to reduce the question of well-foundedness and iterability of $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}, F)$ to that of well-foundedness and iterability of $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}} \,||\, \delta^+, F)$, thereby avoiding any direct use of the frequent extension argument. In what follows we will refer to Zeman [Zem02, Chapter 8] on the core model theory below $0^{\P}$.

So assume $\delta \geq \omega_1$ and the trees $\mathcal{T}$,$\mathcal{U}$ come from the coiteration of $\mathbf{K}^{\mathbf{M}}$ against N. Thus we let $W = \mathbf{K}^{\mathbf{M}}$ in this case; otherwise we keep the rest of the notation as at the beginning of this section. There is a slight non-uniformity which comes into the argument when dealing with the cases $\delta > \omega_1$ and $\delta = \omega_1$.

If $\delta > \omega_1$ the length of the trees $\mathcal{T}$ and $\mathcal{U}$ is δ, as we assume, toward a contradiction, that N out-iterates $\mathbf{K}^{\mathbf{M}} \,||\, \delta$. In the case where $\delta = \omega_1$, our hypothesis reads that $\mathbf{K}^{\mathbf{M}} \,||\, \omega_2$ fails to be universal with respect to countable mice in $\mathbf{V}$, so in this case we assume N iterates past $\mathbf{K}^{\mathbf{M}} \,||\, \omega_2$.

For the rest of the argument, write $\delta^* = \delta$ if $\delta > \omega_1$, and $\delta^* = \omega_2$ if $\delta = \omega_1$. Notice however that even in the case where $\delta = \omega_1$, the extenders $F_{\xi,\xi'}$ are defined as in Lemma 22, that is, using only the trees $\mathcal{T} \upharpoonright \delta$ and $\mathcal{U} \upharpoonright \delta$. Refering back to Lemma 22, pick $\kappa < \lambda$ in C^*. Here $C^* \subseteq \delta$, hence $\lambda < \delta$. Recall also that, consistently with the notation fixed below (8), we write W_α for $M^{\mathcal{T}}_\alpha$ and N_α for $M^{\mathcal{U}}_\alpha$.

Lemma 28. *Let $F^*_{\kappa,\lambda}$ be the extender at (κ, λ) on N_κ derived from $\pi^{\mathcal{U}}_{\kappa,\lambda}$. Then $\tilde{N} = \mathsf{Ult}(N_{\delta^*}, F^*_{\kappa,\lambda})$ is well-founded and iterable.*

It follows that $\mathsf{Ult}(N_{\delta^*} \,||\, \delta^*, F^*_{\kappa,\lambda})$ is well-founded and iterable.

Proof. By the definition of the set C^* there is an iteration map

$$\pi^{\mathcal{U}}_{\kappa,\delta} : N_\kappa \to N_\delta.$$

Write σ for $\pi^{\mathcal{U}}_{\kappa,\delta}$, and use σ to copy the linear iteration tree $\mathcal{U} \upharpoonright [\kappa, \delta^*)$ onto a linear iteration tree $\mathcal{U}'$ on N_δ via the map σ. This is easy to do, as the iteration trees are linear and the extenders used in these trees are internal. We thus obtain a copy map $\sigma' : N_{\delta^*} \to N'_{\delta^*}$ where we write N'_α for $M^{\mathcal{U}'}_\alpha$. Notice that if $\delta = \omega_1$, there may be a truncation on the tree $\mathcal{U} \upharpoonright [\delta, \delta^*)$, but it will not have any negative effect on the argument below. Since σ is Σ^*-preserving, so is σ'. By the choice of κ, the critical points in the tree $\mathcal{U} \upharpoonright [\kappa, \delta^*)$ are at least κ, and the iteration indices are at least $\tau'_\kappa = \kappa^{+N_\kappa}$ (this notation is consistent with that fixed in Lemma 18), as $\mathcal{U} \upharpoonright [\kappa, \delta)$ does not involve any truncation. It is also easy to see that $\tau'_\kappa = \kappa^{+N_{\delta^*}}$. It follows that $N_\kappa \,||\, \tau'_\kappa = N_{\delta^*} \,||\, \tau'_\kappa$ and $\sigma' \upharpoonright (N_{\delta^*} \,||\, \tau'_\kappa) = \sigma \upharpoonright (N_\kappa \,||\, \tau'_\kappa)$. In particular, the extender at (κ, λ) derived from σ' is precisely $F^*_{\kappa,\lambda}$. This tells us that the ultrapower $\mathsf{Ult}(N_{\delta^*}, F^*_{\kappa,\lambda})$ can be embedded into the premouse N'_{δ^*} via the natural factor embedding $k : [a, f]_{F^*_{\kappa,\lambda}} \mapsto \sigma'(f)(a)$ which is Σ^*-preserving, as both σ' and the ultrapower embeddings are. Now N'_{δ^*}, being an iterate of an iterable premouse, is itself iterable, which guarantees

the iterability of $\mathsf{Ult}(N_{\delta^*}, F^*_{\kappa,\lambda})$. The last conclusion of the lemma is an immediate consequence. □

We are now going to use a variant of the "shift lemma" that will allow us to embed $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}} \,||\, \delta^*, F_{\kappa,\lambda})$ into $\mathsf{Ult}(N_{\delta^*} \,||\, \delta^*, F^*_{\kappa,\lambda})$. Given structures Q, Q', a Σ_0-preserving map $\sigma : Q \to Q'$, extenders G at (κ, λ) on Q and G' at (κ, λ') on Q' such that $\sigma[\kappa] \subseteq \kappa$, and an order preserving map $\mathfrak{k} : \lambda \to \lambda'$, we write $(\sigma, \mathfrak{k}) : (Q, G) \to (Q', G')$ if and only if for every $a \in [\lambda]^{<\omega}$ and $x \in \mathcal{P}([\kappa]^{|a|}) \cap Q$,

$$x \in G_a \implies \sigma(x) \cap [\kappa]^{|a|} \in G'_{\mathfrak{k}[a]}.$$

This corresponds to the similar notion discussed in Zeman [Zem02, §2.5]. The slight difference between the notion used here and that in Zeman [Zem02] is that, in our situation, it may happen that $\sigma(\mathsf{cr}(G)) > \mathsf{cr}(G')$. If $(\sigma, \mathfrak{k}) : (Q, G) \to (Q', G')$, then we can run the proof of the shift lemma and show that there is precisely one Σ_0-preserving embedding

$$\sigma' : \mathsf{Ult}(Q, G) \to \mathsf{Ult}(Q', G')$$

such that $\sigma' \upharpoonright \lambda = \mathfrak{k}$ and $\sigma' \circ \pi_G = \pi_{G'} \circ \sigma$, where $\pi_G, \pi_{G'}$ are the corresponding ultrapower embeddings. Moreover, if σ is fully elementary, so is σ'.

Lemma 29. *Let $\kappa < \lambda$ be in C^*, and let $F^*_{\kappa,\lambda}$ be as in Lemma 28. Then the following hold:*

(a) $(\pi^{\mathcal{T}}_{0,\delta^*} \upharpoonright (\mathbf{K}^{\mathbf{M}} \,||\, \delta^*), \pi^{\mathcal{T}}_{0,\delta^*} \upharpoonright \lambda) : (\mathbf{K}^{\mathbf{M}} \,||\, \delta^*, F_{\kappa,\lambda}) \to (N_{\delta^*} \,||\, \delta^*, F^*_{\kappa,\lambda})$.

(b) $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}} \,||\, \delta^*), F_{\kappa,\lambda})$ *is well-founded and iterable.*

Proof. Clause (a) follows by a straigthforward computation using the fact that $\mathsf{cr}(\pi^{\mathcal{T}}_{\lambda,\delta^*}) \geq \lambda$. Clause (b) is then an immediate consequence. □

Now let $S = \mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$, pick θ large regular in $\mathbf{V}$, and let S^θ be as in Lemma 24. If we pick $X \in Y$ such that both $X, Y \in S^\theta$, and let $\kappa = \kappa_X$ and $\lambda = \kappa_Y$, then $F_{\kappa,\lambda} \in \mathbf{M}$. From now on, we work entirely in $\mathbf{M}$. By Lemma 29, $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}} \,||\, \delta^*, F_{\kappa,\lambda})$ is well-founded and iterable. We show that this conclusion can be extended to $\mathbf{K}^{\mathbf{M}}$.

Lemma 30. $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}, F_{\kappa,\lambda})$ *is well-founded and iterable.*

Proof. Assume the contrary. Working in $\mathbf{M}$, let ν be a $\mathbf{K}^{\mathbf{M}}$-cardinal large enough such that, letting $Q = \mathbf{K}^{\mathbf{M}} \,||\, \nu$, the ultrapower $\mathsf{Ult}(Q, F_{\kappa,\lambda})$ is either ill-founded or not iterable, and let $\mathcal{V}$ be a linear iteration tree in $\mathbf{M}$ on Q witnessing this. Again, if $\mathsf{Ult}(Q, F_{\kappa,\lambda})$ is ill-founded, we let $\mathcal{V}$ be the tree of length 1 consisting of this ill-founded ultrapower, in order to treat the two cases uniformly. Let θ^* be a large regular cardinal such that $\mathcal{V} \in H^{\mathbf{M}}_{\theta^*}$, and let Z be an elementary substructure of $H^{\mathbf{M}}_{\theta^*}$ of size less than δ such that $F_{\kappa,\lambda}, \mathcal{V} \in Z$ and $\bar{\delta} = Z \cap \delta \in \delta$. Also, let H be the transitive collapse of Z, and let $\sigma : H \to H^{\mathbf{M}}_{\theta^*}$ be the inverse to the Mostowski collapsing isomorphism. Notice that $\bar{\delta} > \lambda$ and $\sigma(F_{\kappa,\lambda}) = F_{\kappa,\lambda}$. Letting $(\bar{Q}, \bar{\mathcal{V}}) =$

$\sigma^{-1}(Q,\mathcal{V})$, the linear iteration tree $\bar{\mathcal{V}}$ witnesses the ill-foundedness/non-iterability of $\mathsf{Ult}(\bar{Q},F_{\kappa,\lambda})$. We will use the following observation:

Claim 31. *Let $\mathcal{R},\bar{\mathcal{R}}$ be the linear iteration trees on $Q,\bar{Q}$ respectively, coming from the comparison of Q against $\bar{Q}$. Then the critical points in $\bar{\mathcal{R}}$ are above $\bar{\delta}$.*

Proof. If $\bar{\delta}$ is not overlapped by an extender in $\bar{Q}$ then this is immediate. So suppose $\bar{\delta}$ is overlapped. Here we apply a condensation result particular for the indexing of the extenders we are using here; recall this is the indexing described in Zeman [Zem02, Chapter 8]. It follows that there is $\mu<\bar{\delta}$ and $\beta>\delta$ such that $E^{\bar{Q}}_{\beta}=\varnothing$, $\mathsf{cr}(E^{\bar{Q}}_{\alpha})=\mu$ whenever $\mu^{+\bar{Q}}\leq\alpha<\beta$ and α indexes an extender in $\bar{Q}$, and for every $\alpha>\beta$ that index an extender in $\bar{Q}$ we have $\mathsf{cr}(E^{\bar{Q}}_{\alpha})>\beta$.

Since $\sigma:\bar{Q}\to Q$ is elementary and $\mathsf{cr}(\sigma)=\bar{\delta}$ then, by Zeman [Zem02, Lemma 8.2.2], $E^{\bar{Q}}\restriction\beta=E^{Q}\restriction\beta$. It follows that in the coiteration of Q against $\bar{Q}$, all iteration indices used in $\bar{\mathcal{R}}$ are larger than β, but by the discussion above, also the corresponding critical points are above β, and hence above $\bar{\delta}$. □

We can now complete the proof of the lemma. Since $\delta^*\geq\omega_2^{\mathbf{M}}$, the initial segment $\mathbf{K}^{\mathbf{M}}$ is universal for all premice in $\mathbf{M}$ of size less than δ^*. Hence $\mathbf{K}^{\mathbf{M}}\,||\,\delta^*$ iterates past $\bar{Q}$, and the coiteration terminates after less than δ^* steps. Let $\mathcal{R}'$ be the linear iteration tree on the $\mathbf{K}^{\mathbf{M}}\,||\,\delta^*$-side of the coiteraton of $\bar{Q}$ against $\mathbf{K}^{\mathbf{M}}\,||\,\delta^*$. Since $\mathbf{K}^{\mathbf{M}}\,||\,\delta^*\trianglelefteq Q$ and δ^* is a regular cardinal, the iteration tree on the $\bar{Q}$-side of this coiteration is precisely $\bar{\mathcal{R}}$, and the iteration tree $\mathcal{R}'$ is essentially the same as $\mathcal{R}$ with the only difference that $M^{\mathcal{R}'}_{\alpha}=M^{\mathcal{R}}_{\alpha}\,||\,\delta^*$ for all $\alpha<\mathsf{lh}(\mathcal{R})$. By the above discussion, there is an iteration map $\pi^{\bar{\mathcal{R}}}_{\infty}:\bar{Q}\to M^{\bar{\mathcal{R}}}_{\infty}$ along the main (and only) branch of the iteration tree $\bar{\mathcal{R}}$, where $M^{\bar{\mathcal{R}}}_{\infty}$ is the last model on $\bar{\mathcal{R}}$. Moreover, $M^{\bar{\mathcal{R}}}_{\infty}\trianglelefteq M^{\mathcal{R}'}_{\infty}$. By the Claim, $\mathsf{cr}(\pi^{\bar{\mathcal{R}}}_{\infty})\geq\bar{\delta}>\lambda$. It follows that $\mathsf{Ult}(\bar{Q},F_{\kappa,\lambda})$ can be embedded into $\mathsf{Ult}(M^{\bar{\mathcal{R}}}_{\infty},F_{\kappa,\lambda})$ by $[a,f]_{F_{\kappa,\lambda}}\mapsto[a,\pi^{\bar{\mathcal{R}}}_{\infty}(f)]$, which shows that $\mathsf{Ult}(M^{\bar{\mathcal{R}}}_{\infty},F_{\kappa,\lambda})$ is either ill-founded, or else not iterable. It follows that $\mathsf{Ult}(M^{\mathcal{R}'}_{\infty},F_{\kappa,\lambda})$ is ill-founded or not iterable.

On the other hand, $Q^*=\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta^*,F_{\kappa,\lambda})$ is well-founded and iterable by Lemma 29.(b). Let ρ be the associated ultrapower map. Use ρ to copy the linear iteration tree $\mathcal{R}'$ onto a linear iteration tree $\mathcal{R}^*$ on Q^* via the map ρ. Let $\rho^*:M^{\mathcal{R}'}_{\infty}\to M^{\mathcal{R}^*}_{\infty}$ be the copy map between the last two models in the copy construction. Since the iteration indices of $\mathcal{R}'$ are above $\bar{\delta}$, the map ρ^* agrees with ρ below $\bar{\delta}$. It follows that the extender on $M^{\mathcal{R}'}_{\infty}$ at (κ,λ) derived from ρ^* is precisely $F_{\kappa,\lambda}$, hence $\mathsf{Ult}(M^{\mathcal{R}'}_{\infty},F_{\kappa,\lambda})$ can be embedded into $M^{\mathcal{R}^*}_{\infty}$ via the natural map $[a,f]_{F_{\kappa,\lambda}}\mapsto\rho^*(f)(a)$. As $\mathcal{R}^*_{\infty}$, being an iterate of an iterable premouse, is itself iterable, this proves the iterability of $\mathsf{Ult}(M^{\mathcal{R}'}_{\infty},F_{\kappa,\lambda})$. Now we have have a contradiction with the conclusion

of the previous paragraph, thereby completing the proof of Lemma 30 and, therefore, of Theorem 5.(b). $\square$

4. Discussion

In this section we discuss some issues concerning why we are unable to prove our main result in its full generality, and outline a possible approach that may lead to removing all our restricting assumptions, at least in the absence of Woodin cardinals. Our conjecture thus can be formulated as follows:

Conjecture 32. *Assume that there is no proper class inner model with a Woodin cardinal. Let δ be an uncountable regular cardinal, and let $\mathbf{M}$ be a proper class inner model such that $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$ is stationary. Then the following hold:*

(a) *If $\delta > \omega_1$, then $\mathbf{K}^{\mathbf{M}} \,||\, \delta$ is universal for all iterable premice in $\mathbf{V}$ of cardinality less than δ.*
(b) *If $\delta = \omega_1$, then $\mathbf{K}^{\mathbf{M}} \,||\, \omega_2$ is universal for all countable iterable premice in $\mathbf{V}$.*

Theorem 5.(b) says that the above conclusions (a) and (b) hold under the rather restrictive anti-large cardinal hypothesis that a sharp for an inner model with a strong cardinal does not exist. Unfortunately, we do not know how to extend our proof of Theorem 5.(b) to the more general situation with no inner models with Woodin cardinals. We do, however, have some partial results along these lines that we proceed to describe.

4.1. **The case $\delta > \omega_1$.** One drawback of our proof of Theorem 5.(a) is the direct use of the frequent extension argument. As we have already explained above, the use of this argument requires us to work with the set S_δ from (1) in place of $\mathcal{P}_\delta(\delta^+) \cap \mathbf{M}$, as it relies on the fact that countable subsets of structures in S_δ can be internally covered. And, as explained before, there is no way around, by the results in Räsch-Schindler [RS05] and references therein. Our proof of Theorem 5.(b) avoids the direct use of the frequent extension argument by showing that for suitable δ and X, Y, (i) $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}} \,||\, \delta, F_{X,Y})$ is well-founded and iterable, and (ii) the question of well-foundedness and iterability of $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}, F_{X,Y})$ can be reduced to that of the well-foundedness and iterability of $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}} \,||\, \delta, F_{X,Y})$. Here, (ii) relies on the result from Schimmerling-Steel [SS99] that $\mathbf{K}^{\mathbf{M}} \,||\, \delta$ is universal for iterable premice of size less than δ in $\mathbf{M}$ whenever $\delta \geq \omega_2^{\mathbf{M}}$. Under the assumption "no proper class inner model with a Woodin cardinal", (ii) needs to be replaced with a stronger statement, namely that $\mathsf{Ult}(W, F_{X,Y})$ is well-founded and the phalanx $(W, \mathsf{Ult}(W, F_{X,Y}), \kappa_Y)$ is iterable, where W is the extender model witnessing the soundness of a suitably long initial segment of $\mathbf{K}^{\mathbf{M}}$.

In the following, we outline a way of generalizing the approach from the proof of Theorem 5.(b) for $\delta > \omega_1$ to the situation where we assume no

proper class inner models with a Woodin cardinal. The argument from the proof of Theorem 5.(b) can be generalized to obtain the following:

$$\begin{array}{l}\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta, F_{X,Y}) \text{ is well-founded, and} \\ (\mathbf{K}^{\mathbf{M}}\,||\,\delta, \mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta, F_{X,Y}), \kappa_Y) \text{ is iterable.}\end{array} \tag{14}$$

However, we do not know how to use (14) along with the universality of $\mathbf{K}^{\mathbf{M}}\,||\,\delta$ to prove the iterability of the phalanx $(W, \mathsf{Ult}(W, F_{X,Y}), \kappa_Y)$.

The conclusion (14) is established using a copying construction that generalizes that in the proof of Theorem 5.(b). More precisely, letting $\kappa = \kappa_X$, if $\mathcal{U}$ is the normal iteration tree on N coming from the coiteration with W as described below (2), we can "embed" $\mathcal{U} \restriction \kappa$ into $\mathcal{U}$ using a system of maps σ_i where $\sigma_\alpha : M^{\mathcal{U}}_\alpha \to M^{\mathcal{U}}_\alpha$ is the identity map for $\alpha < \kappa$, and $\sigma_\kappa : M^{\mathcal{U}}_\kappa \to M^{\mathcal{U}}_\delta$ is the iteration map $\pi^{\mathcal{U}}_{\kappa,\delta}$. A straigthforward, but a bit tedious computation shows that we can copy the iteration tree $\mathcal{U}$ onto a normal iteration tree $\mathcal{U}'$ of length $\delta + \delta$ extending $\mathcal{U}$, where the copying construction uses the maps σ_i instead of a single map. The point here is to verify that there are no "conflicts" when we copy, that is, $\mathcal{U}'$ is indeed a normal iteration tree. The situation can be then depicted by the following diagram; we write N_α for $M^{\mathcal{U}}_\alpha$ and N'_α for $M^{\mathcal{U}'}_\alpha$.

$$\begin{tikzcd}
\mathcal{U}' & N \arrow[r, dotted] & N'_h \arrow[r, dotted] & N'_\kappa \arrow[r, dotted] & N'_\delta \arrow[r, dotted] & N'_{\delta+\alpha} \arrow[r, dotted] & N'_{\delta+\delta} & \\
\mathcal{U} & N \arrow[r, dotted] \arrow[u, "\sigma_0 = \mathsf{id}"] & N_h \arrow[r, dotted] \arrow[u, "\sigma_h = \mathsf{id}"] & N_\kappa \arrow[r, dotted] \arrow[ur, "\sigma_\kappa = \pi^{\mathcal{U}}_{\kappa,\delta}"] & N_\delta \arrow[r, dotted] & N_{\kappa+\alpha} \arrow[r, dotted] \arrow[u, "\sigma_{\kappa+\alpha}"] & N_\delta \arrow[r, dashed, "\pi_{F^*_{X,Y}}"] \arrow[u, "\sigma_\delta"] & \tilde{N} \arrow[ul, "\tilde{\sigma}"']
\end{tikzcd}$$

As the iteration indices in the coinitial segment of $\mathcal{U} \restriction [\kappa, \delta)$ are larger than κ, the copy map σ_δ agrees with σ_κ on $\mathcal{P}(\kappa) \cap N_\kappa = \mathcal{P}(\kappa) \cap N_\delta$. It follows that the two derived extenders agree, and in fact they agree with the map $\sigma_X \restriction (\mathcal{P}(\kappa) \cap N_\delta)$. (See the settings above (5)). So if we pick Y such that $X \in Y$, and let $\lambda = \kappa_Y$ and $F^*_{X,Y}$ be the (κ, λ)-extender derived from $\sigma_{X,Y} \restriction (\mathcal{P}(\kappa) \cap N_\delta)$, we see that $\tilde{N} = \mathsf{Ult}(N_\delta, F^*_{X,Y})$ is well-founded, and the factor map $\tilde{\sigma} : \tilde{N} \to N'_{\delta+\delta}$ is the identity below λ, and sends λ to δ.

It follows that the pair $(\mathsf{id}, \tilde{\sigma})$ is an embedding of the phalanx $(N'_{\delta+\delta}, \tilde{N}, \delta)$ into $N'_{\delta+\delta}$, which proves the iterability of the phalanx. This of course yields the iterability of the phalanx $(N'_{\delta+\delta}\,||\,\delta, \tilde{N}\,||\,\delta, \lambda)$. Recall that $\mathcal{T}$ is the iteration tree on the W-side of the coiteration of N with W; see again the settings below (2). As in the proof of Theorem 5.(b), we can then show that the phalanx $(\mathbf{K}^{\mathbf{M}}\,||\,\delta, \mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta, F_{X,Y}), \lambda)$ can be elemantarily embedded into $(N'_{\delta+\delta}\,||\,\delta, \tilde{N}\,||\,\delta, \lambda)$ using the pair of maps $(\pi^{\mathcal{T}}_\delta \restriction (\mathbf{K}^{\mathbf{M}}\,||\,\delta), k)$ where $k : \mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta, F_{X,Y}) \to \tilde{N}\,||\,\delta$ is the unique elementary map that agrees with $\pi^{\mathcal{T}}_{0,\delta}$ below λ and satisfies $k \circ \pi_{X,Y} = \pi^*_{X,Y} \circ \pi^{\mathcal{T}}_{0,\delta}$, and $\pi^*_{X,Y}$ and $\pi_{X,Y}$ are the ultrapower embeddings coming from $\mathsf{Ult}(N_\delta\,||\,\delta, F^*_{X,Y})$ and

$\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta, F_{X,Y})$, respectively. This completes the sketch of the proof of (14).

4.2. **The case $\delta = \omega_1$.** Notice that this case is new only if $\delta = \omega_1^M$, as otherwise it falls under the discussion above. Although superficially this case may appear to be very special, it is interesting in many respects. The main difference between this case and the one discussed above is that, as we already mentioned in the introduction, we cannot expect $\mathbf{K}^{\mathbf{M}}\,||\,\delta$ to be universal for countable mice, by the result of Jensen [Jena]. In the proof of Theorem 5.(b), we still follow the strategy outlined above, that is, we first establish the well-foundedness and iterability of $\mathbf{K}^{\mathbf{M}}\,||\,\delta^+$, and then use the universality of this initial segment of $\mathbf{K}^{\mathbf{M}}$ to lift the conclusion to $\mathbf{K}^{\mathbf{M}}$. The new element here involves using the coiteration of W against N of length δ^+. However, the extender $F_{X,Y}$ is determined already by the coiteration of length δ. Once we are in the situation below a sharp for an inner model with a strong cardinal, all iteration trees can be made linear, so once we establish the well-foundedness of $\mathsf{Ult}(N_{\delta^+}, F^*_{X,Y})$ we easily embed $\mathsf{Ult}(\mathbf{K}^{\mathbf{M}}\,||\,\delta^+, F_{X,Y})$ into $\mathsf{Ult}(N_{\delta^+}, F^*_{X,Y})$ since $\pi^{\mathcal{T}}_{0,\delta^+} = \pi^{\mathcal{T}}_{\delta,\delta^+} \circ \pi^{\mathcal{T}}_{0,\delta}$ and the critical point of $\pi^{\mathcal{T}}_{\delta,\delta^+}$ is at least δ. In the situation with many strong cardinals, we have non-linear iteration trees, and although the argument outlined in the above case where $\delta > \omega_1$ proves that $(N_{\delta^+}, \mathsf{Ult}(N_{\delta^+}, F^*_{X,Y}), \kappa_Y)$ is iterable, the argument for linear iterations can be mimicked here only if we know that δ is on the branch $[0, \delta^+]_{\mathcal{T}}$, or, more generally, if some element of the set C^* defined below Lemma 17 is on that branch. It is also easy to see that if δ is not on the branch $[0, \delta^+]_{\mathcal{T}}$ then there is a sharp for an inner model with a strong cardinal, but we do not know if a substantially stronger large cardinal hypothesis can be extrected in this situation. (For instance an inner model with two strong cardinals.)

4.3. **No inner models of M with Woodin cardinals.** Finally, it is natural to wonder whether the anti-large cardinal assumption in Theorem 5.(a) that there are no inner models with Woodin cardinals can be localized to $\mathbf{M}$.

In this setting, we do not know how to prove iterability in the sense of $\mathbf{V}$ of proper class extender models of $\mathbf{M}$, and we do not even know if such models are iterable in $\mathbf{V}$. We can however prove a weaker form of iterability for such models, and establish a weak form of our main theorem. The problem is that this weak form does not seem to suffice to deduce any interesting applications, so we omit its proof. The precise statement of the result we have in this case is as follows:

Theorem 33. *Assume that $\mathbf{M}$ is a proper class inner model, and $\delta > \omega_1$ is a regular cardinal in $\mathbf{V}$ such that S_δ is stationary. Granting that in $\mathbf{M}$ there is no proper class inner model with a Woodin cardinal, the initial segment $\mathbf{K}^{\mathbf{M}}\,||\,\delta$ is universal for all iterable 1-small premice in $\mathbf{V}$ of cardinality less than δ in the following weak sense:*

If N is a 1*-small iterable premouse in* $\mathbf{V}$ *of cardinality less than* δ*, then there is a pair of iteration trees* $(\mathcal{T},\mathcal{U})$ *of length less than* δ*, where* $\mathcal{T}$ *is on* $\mathbf{K}^{\mathbf{M}}\,||\,\delta$ *and* $\mathcal{U}$ *is on* N*, such that* $\mathcal{U}$ *has a last model* $M^{\mathcal{U}}_\infty$*, there is no truncation on the main branch of* $\mathcal{U}$*, and one of the following holds:*

(a) *The tree* $\mathcal{T}$ *has a last well-founded model* $M^{\mathcal{T}}_\infty$*, and* $M^{\mathcal{U}}_\infty$ *is an initial segment of* $M^{\mathcal{T}}_\infty$*.*

(b) *The tree* $\mathcal{T}$ *has a limit length, does not have a cofinal well-founded branch, and* $\delta(\mathcal{T}) = \delta(\mathcal{U})$*. Moreover, no ordinal larger than* $\delta(\mathcal{U})$ *indexes an extender on* $M^{\mathcal{U}}_\infty$ *and, in some generic extension of* $\mathbf{V}$*, there is a cofinal branch* b *through* $\mathcal{T}$ *such that the direct limit along* b *has well-founded part of length at least* $\mathbf{On} \cap M^{\mathcal{U}}_\infty$*.*

The interest here, of course, would be to replace this weak universality with its genuine version or, if this is not possible, to see whether this version can actually be useful. For example, this version seems irrelevant for Σ^1_3-correctness, for which one needs to assume that there are no inner models with a Woodin cardinal *in* $\mathbf{V}$.

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Andrés E. Caicedo, Boise State University, Department of Mathematics, 1910 University Drive, Boise, ID 83725

E-mail address: `caicedo@math.boisestate.edu`

URL: `http://math.boisestate.edu/~caicedo`

Martin Zeman, University of California at Irvine, Department of Mathematics, Irvine, CA 92697

E-mail address: `mzeman@math.uci.edu`

URL: `http://www.math.uci.edu/~mzeman/`